COMPACTA WITH SHAPES OF FINITE COMPLEXES: A DIRECT APPROACH TO THE EDWARDS-GEOGHEGAN-WALL OBSTRUCTION

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Abstract. An important "stability" theorem due to D.A. Edwards and R. Geoghegan characterizes those compacta having the same shape as a finite CW complex. In this note we present straightforward and self-contained proof of that theorem.

1. Introduction

In the 1970's D.A. Edwards and R. Geoghegan solved two fundamental problems in shape theory—both related to the issue of "stability". Roughly speaking, these problems ask when a "bad" space has the same shape as a "good" space. For simplicity, we focus on the following versions of these problems:

Problem A. *Give necessary and sufficient conditions for a connected finite dimensional compactum Z to have the pointed shape of a CW complex.*

Problem B. *Give necessary and sufficient conditions for a connected finite dimensional compactum Z to have the pointed shape of a finite CW complex.*

Solutions to these problems can be found in the sequence of papers: [EG1], [EG2], [EG3]. One particularly nice version of the solution to Problem A states: *Z has the pointed shape of a CW complex if and only if each of its homotopy pro-groups is stable.* By applying this result and C.T.C. Wall's famous work on finite homotopy types [Wa], Edwards and Geoghegan were then able to solve Problem B: *Z has the pointed shape of a finite CW complex if and only if each of its homotopy pro-groups is stable and an intrinsically defined Wall obstruction $\omega(Z,z) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(Z,z)])$ vanishes.*

In order to understand Edwards and Geoghegan's solution to Problem B, it is then necessary to understand two things—the solution to Problem A, and Wall's work on the finiteness obstruction. Since both of these tasks are substantial—and since Problem B can arise quite naturally without regards to Problem A—we became interested in finding a simpler and more direct solution to Problem B. This note contains that solution. It may be viewed as a sequel to [Ge], where Geoghegan presented a new and more elementary solution to Problem A. In the same spirit, we feel that our work offers a simplified view of Problem B.

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The strategy we use in attacking Problem B is straightforward and very natural. Given a connected n-dimensional pointed compactum Z, begin with an inverse system $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ of finite n-dimensional (pointed) complexes with (pointed) cellular bonding maps that represents Z. Under the assumption that pro-π_k is stable for all k, we borrow a technique from [Fe] allowing us to attach cells to the K_i's so that the bonding maps induce π_k-isomorphisms for increasingly large k. Our goal then is to reach a *finite* stage where the bonding maps induce π_k-isomorphisms for all k, and are therefore homotopy equivalences. This would imply that Z has the shape of a finite complex. As expected, we confront an obstruction lying in the reduced projective class group (or "Wall group") of pro-π_1. Instead of invoking theorems from [Wa], we uncover this obstruction in the natural context of the problem at hand. (In fact, the main result of [Wa] can then be obtained as a corollary.) Another advantage to our approach is that all CW complexes used in this paper are finite. This makes both the algebra and the shape theory more elementary.

2. Background

In this section we provide some background information on inverse sequences and shape theory. We include only what is needed in this paper. More general treatments of these topics can be found in [Ge]. In addition, we will review the definition of a reduced projective class group.

2.1. **Inverse sequences.**

Let $\mathcal{C}$ be a category and let

$$C_0 \xleftarrow{\lambda_1} C_1 \xleftarrow{\lambda_2} C_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of objects and morphisms from $\mathcal{C}$. A *subsequence* of $\{C_i, \lambda_i\}$ is an inverse sequence of the form

$$C_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} C_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} C_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots .$$

In the future we will denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \le j)$ by $\lambda_{i,j}$.

We say that sequences $\{C_i, \lambda_i\}$ and $\{D_i, \mu_i\}$ are *pro-equivalent* if, after passing to subsequences, there exists a commuting diagram:

$$\begin{array}{ccccccccccc}
C_{i_0} & & \xleftarrow{\lambda_{i_0+1,i_1}} & & C_{i_1} & & \xleftarrow{\lambda_{i_1+1,i_1}} & & C_{i_2} & & \xleftarrow{\lambda_{i_2+1,i_3}} & \cdots \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & \swarrow & , \\
& & D_{j_0} & & \xleftarrow{\mu_{i_0+1,i_1}} & & D_{j_1} & & \xleftarrow{\mu_{i_1+1,i_1}} & & D_{j_2} & & \cdots
\end{array}$$

where the diagonal arrows denote morphisms from $\mathcal{C}$. Clearly an inverse sequence is pro-equivalent to any of its subsequences. To avoid tedious notation, we frequently do not distinguish $\{C_i, \lambda_i\}$ from its subsequences. Instead we simply assume that $\{C_i, \lambda_i\}$ has the desired properties of a preferred subsequence—often prefaced by the words "after passing to a subsequence and relabelling".

An inverse sequence $\{C_i, \lambda_i\}$ is *stable* if it is pro-equivalent to a constant sequence

$$D \xleftarrow{id} D \xleftarrow{id} D \xleftarrow{id} \cdots .$$

For example, if each λ_i is an isomorphism from $\mathcal{C}$, it is easy to show that $\{C_i, \lambda_i\}$ is stable.

When the objects of the category $\mathcal{C}$ are sets and the morphisms are functions, we may define the *inverse limit* of a sequence $\{C_i, \lambda_i\}$ by

$$\varprojlim \{C_i, \lambda_i\} = \left\{ (c_0, c_1, c_2, \cdots) \in \prod_{i=0}^{\infty} C_i \,\middle|\, \lambda_i(c_i) = c_{i-1} \text{ for all } i \geq 1 \right\}.$$

Notice that for each i, there is a *projection morphism* $p_i : \varprojlim \{C_i, \lambda_i\} \to C_i$.

2.2. **Inverse sequences of groups.** Of particular interest to us is the category $\mathcal{G}$ of groups and group homomorphisms. It is easy to see that an inverse sequence of groups $\{G_i, \lambda_i\}$ is stable if and only if, after passing to a subsequence and relabelling, there is a commutative diagram of the form

$$\begin{array}{ccccccccc} G_0 & \xleftarrow{\lambda_1} & G_1 & \xleftarrow{\lambda_2} & G_2 & \xleftarrow{\lambda_3} & G_3 & \xleftarrow{\lambda_4} & \cdots \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & \nwarrow \swarrow & & & \\ & im(\lambda_1) & \xleftarrow{\cong} & im(\lambda_2) & \xleftarrow{\cong} & im(\lambda_3) & \xleftarrow{\cong} & \cdots & \end{array},$$

where all unlabeled maps are restrictions of the λ_i's. In this case $\varprojlim \{C_i, \lambda_i\} \cong im(\lambda_i)$ and each projection homomorphism takes $\varprojlim \{C_i, \lambda_i\}$ isomorphically onto the corresponding $im(\lambda_i)$.

The sequence $\{G_i, \lambda_i\}$ is *semistable* (or *Mittag-Leffler*) if it is pro-equivalent to an inverse sequence $\{H_i, \mu_i\}$ for which each μ_i is surjective. Equivalently, $\{G_i, \lambda_i\}$ is semistable if, after passing to a subsequence and relabelling, there is a commutative diagram of the form

$$\begin{array}{ccccccccc} G_0 & \xleftarrow{\lambda_1} & G_1 & \xleftarrow{\lambda_2} & G_2 & \xleftarrow{\lambda_3} & G_3 & \xleftarrow{\lambda_4} & \cdots \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & \nwarrow \swarrow & & & \\ & im(\lambda_1) & \twoheadleftarrow & im(\lambda_2) & \twoheadleftarrow & im(\lambda_3) & \twoheadleftarrow & \cdots & \end{array},$$

where the symbol "$\twoheadleftarrow$" denotes a surjection.

2.3. **Inverse sequences of CW complexes.** Another important category $\mathcal{FH}_0$ consists pointed connected finite CW complexes and pointed homotopy classes of maps. Recall that a space is *pointed* if a preferred basepoint has been chosen, while a map between pointed spaces is *pointed* if basepoint is taken to basepoint. We will frequently refer to pointed spaces and maps without explicitly mentioning the basepoints. We will refer to an inverse sequence constructed from $\mathcal{FH}_0$ as a *tower of finite complexes.*

The *dimension,* $\dim(\{K_i, f_i\})$, of a tower of finite complexes is the supremum (possibly ∞) of the dimensions of the K_i's. The *homotopy dimension* of $\{K_i, f_i\}$ is defined by:

$$\operatorname{hom}\dim(\{K_i, f_i\}) = \inf\{\dim\{L_i, g_i\} \mid \{L_i, g_i\} \text{ is pro-equivalent to } \{K_i, f_i\}\}.$$

Given a tower of finite complexes $\{K_i, f_i\}$ and an integer $k \geq 1$, there is a corresponding inverse sequence of groups:

$$\pi_k(K_0) \xleftarrow{f_{1*}} \pi_k(K_1) \xleftarrow{f_{2*}} \pi_k(K_2) \xleftarrow{f_{3*}} \cdots .$$

It is easy to see that pro-equivalent towers of complexes yield pro-equivalent sequences of groups. The pro-equivalence class will be denoted *pro*-$\pi_k(\{K_i, f_i\})$ and the inverse limit (which is well defined up to isomorphism) will be denoted $\check{\pi}_k(\{K_i, f_i\})$.

2.4. **Shapes of compacta.** Let Z be a compact, connected, metric space with basepoint z. We may associate to Z a tower of finite complexes as follows. First choose a sequence $\{\mathcal{U}_i\}_{i=o}^{\infty}$ of finite open covers of Z such that each $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$, $mesh(\mathcal{U}_i) \to 0$, and each $\mathcal{U}_i$ has a unique element U_i^* containing z Then, for each i, let N_i be the nerve of $\mathcal{U}_i$ with basepoint corresponding to U_i^*. This gives us a tower of finite complexes $\{N_i, g_i\}$ where each $g_i : N_i \to N_{i-1}$ is the naturally induced simplicial map. It is straightforward to show that any two towers constructed in this manner are pro-equivalent. We say that Z and Z' have the same *pointed shape* if their corresponding towers are pro-equivalent. The *shape dimension* of Z is defined to be the homotopy dimension of a corresponding tower. Clearly, the shape dimension of Z is less than or equal to its topological dimension.

Another way to associate a tower of finite complexes to Z is to realize Z as the inverse limit of a sequence of pointed connected finite complexes. This inverse sequence will be pro-equivalent to the one obtained above, and thus may be used in its place. See [Mo] for details.

Given a pointed, compact, connected, metric space Z and an associated tower $\{K_i, f_i\}$, define *pro*-$\pi_k(Z)$ to be *pro*-$\pi_k(\{K_i, f_i\})$ and the *Čech homotopy group* $\check{\pi}_k(Z)$ to be $\check{\pi}_k(\{K_i, f_i\})$.

2.5. **The reduced projective class group.** If Λ is a ring, we say that two finitely generated projective Λ-modules P and Q are *stably equivalent* if there exist finitely generated free Λ-modules F_1 and F_2 such that $P \oplus F_1 \cong Q \oplus F_2$. Under the operation of direct sum, the stable equivalence classes of finitely generated projective modules form a group $\widetilde{K}_0(\Lambda)$ known as the *reduced projective class group* (or *Wall group*) of Λ. In this group, a finitely generated projective Λ-module P represents the trivial element if and only if it is *stably free*, i.e., there exists a finitely generated free Λ-module F such that $P \oplus F$ is free.

3. Main Results

We are now ready to state and prove the main results of this paper.

Theorem 3.1. *Let $\{K_i, f_i\}$ be a tower of finite complexes having finite homotopy dimension and stable pro-π_k for all k. Then there is a well defined obstruction $\omega(\{K_i, f_i\}) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{K_i, f_i\}]))$ which vanishes if and only if $\{K_i, f_i\}$ is stable.*

Translating Theorem 3.1 into the language of shape theory yields the desired solution to Problem B:

Theorem 3.2. *A connected compactum Z with finite shape dimension has the pointed shape of a finite CW complex if and only if each of its homotopy pro-groups is stable and an intrinsically defined Wall obstruction $\omega(Z) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(Z)])$ vanishes.*

Our proof begins with two lemmas. The first is a simple and well known algebraic observation.

Lemma 3.3. *Let C_* be a chain complex of finitely generated free Λ-modules, and suppose that $H_i(C_*) = 0$ for $i \leq k$. Then*

1. $\ker \partial_i$ *is finitely generated and stably free for all $i \leq k+1$, and*
2. $H_{k+1}(C_*)$ *is finitely generated.*

Proof. For the first assertion, begin by noting that $\ker \partial_0 = C_0$ is finitely generated and free. Proceeding inductively for $j \leq k+1$, assume that $\ker \partial_{j-1}$ is finitely generated and stably free. Since $H_{j-1}(C_*)$ is trivial, we have a short exact sequence

$$0 \to \ker \partial_j \to C_j \to \ker \partial_{j-1} \to 0.$$

By our assumption on $\ker \partial_{j-1}$, the sequence splits. Therefore, $\ker \partial_j \oplus \ker \partial_{j-1} \cong C_j$, which implies that $\ker \partial_j$ is finitely generated and stably free.

The second assertion follows from the first since $H_{k+1}(C_*) = \ker \partial_{k+1} \diagup im \partial_{k+2}$. $\square$

The second lemma—which is really the starting point of our proof of Theorem 3.1—was extracted from [Fe, Th. 4]. It utilizes the following standard notation and terminology. For a map $f : K \to L$, the mapping cylinder of f will be denoted $M(f)$. The relative homotopy and homology groups of the pair $(M(f), K)$ will be abbreviated to $\pi_i(f)$ and $H_i(f)$. We say that f is *k-connected* if $\pi_i(f) = 0$ for all $i \leq k$; or equivalently, $f_* : \pi_i(K) \to \pi_i(L)$ is an isomorphism for $i < k$ and a surjection when $i = k$. The universal cover of a space K will be denoted $\widetilde{K}$. If $f : K \to L$ induces a π_1-isomorphism, then $\widetilde{f} : \widetilde{K} \to \widetilde{L}$ denotes a lift of f.

Lemma 3.4 (The Tower Improvement Lemma)**.** *Let $\{K_i, f_i\}$ be a tower of finite complexes with stable pro-π_k for $k \leq n$ and semistable pro-π_{n+1}. Then there is a pro-equivalent tower $\{L_i, g_i\}$ of finite complexes with the property that each g_i is $(n+1)$-connected. Moreover, after passing to a subsequence of $\{K_i, f_i\}$ and relabelling, we may assume that:*

1. *each L_i is constructed from K_i by inductively attaching finitely many k-cells for $2 \leq k \leq n+2$,*
2. *each g_i is an extension of f_i such that $g_i(K_i \cup (\text{new cells of dimension } \leq k)) \subset (K_{i-1} \cup (\text{new cells of dimension } \leq k-1))$.*

Proof. Our proof is by induction on n.

Step 1. $(n = 0)$ Let $\{K_i, f_i\}$ be a tower with semistable pro-π_1. By attaching 2-cells to the K_i's, we wish to obtain a new tower in which all bonding maps induce surjections on π_1.

By semistability, we may (by passing to a subsequence and relabelling) assume that each f_{i*} maps $f_{i+1*}(\pi_1(K_{i+1}))$ onto $f_i(\pi_1(K_i))$. Let $\{{}^i g_j\}_{j=1}^{N_i}$ be a finite generating set for $\pi_1(K_i)$ and for each ${}^ig_j$ choose ${}^ih_j \in f_{i+1*}(\pi_1(K_{i+1}))$ such that $f_{i*}\left({}^ig_j\right) = f_{i*}\left({}^ih_j\right)$. For each element of the form ${}^ig_j\left({}^ih_j\right)^{-1} \in \pi_1(K_i)$, attach a 2-cell to K_i which kills that element. Call the resulting complexes L_i's, and note that each f_i extends to a map $k_i : L_i \to K_{i-1}$. Define $g_i : L_i \to L_{i-1}$ to be k_i composed with the inclusion $K_{i-1} \hookrightarrow L_{i-1}$. This leads to the following commutative diagram:

$$\begin{tikzcd}
K_0 & & K_1 \arrow[ll, "f_1"] \arrow[dl] & & K_2 \arrow[ll, "f_2"] \arrow[dl] & & K_3 \arrow[ll, "f_3"] \arrow[dl] & \cdots \arrow[l, "f_4"] \\
& L_1 \arrow[ul, "k_1"] & & L_2 \arrow[ul, "k_2"] \arrow[ll, "g_2"] & & L_3 \arrow[ul, "k_3"] \arrow[ll, "g_3"] & & \cdots \arrow[ll, "g_4"]
\end{tikzcd}$$

which ensures that the tower $\{L_i, g_i\}$ is pro-equivalent to the original.

Note that each $g_{i+1*} : \pi_1(L_{i+1}) \to \pi_1(L_i)$ is surjective. Indeed, the loops in K_i corresponding to the generating set $\{{}^ig_j\}$ of $\pi_1(K_i)$ still generate $\pi_1(L_i)$; moreover, in $\pi_1(L_i)$ each ${}^ig_j$ becomes identified with ${}^ih_j$ which lies in $im\,(g_{i+1*})$. Properties 1 and 2 are immediate from the construction.

Step 2. $(n > 0)$ Now suppose $\{K_i, f_i\}$ is a tower such that pro-π_k is stable for all $k \leq n$ and pro-π_{n+1} is semistable.

We may assume inductively that there is an equivalent tower $\{L_i', g_i'\}$ which has n-connected bonding maps and (after passing to a subsequence of $\{K_i, f_i\}$ and relabelling) satisfies:

1′. each L_i' is constructed from K_i by inductively attaching finitely many k-cells for $2 \leq k \leq n+1$, and
2′. each g_i' is an extension of f_i such that $g_i'\left(K_i \cup (\text{new cells of dimension } \leq k)\right) \subset \left(K_{i-1} \cup (\text{new cells of dimension } \leq k-1)\right)$.

Since pro-π_{n+1} is semistable, we may also assume that:

3′. g_{i*}' maps $g_{i+1*}'(\pi_{n+1}(L_{i+1}'))$ onto $g_{i*}'(\pi_{n+1}(L_i'))$ for all i.

Since the g_i''s are n-connected, then each $g_{i*}' : \pi_k\left(L_i'\right) \to \pi_k\left(L_{i-1}'\right)$ is an isomorphism for $k < n$. In addition, each $g_{i*}' : \pi_n\left(L_i'\right) \to \pi_n\left(L_{i-1}'\right)$ is surjective; but since pro-π_n is stable, all but finitely many of these surjections must be isomorphisms. So, by dropping finitely many terms and relabelling, we assume that these also are isomorphisms.

Our goal is now clear—by attaching $(n+2)$-cells to the L_i''s, we wish to make each bonding map $(n+1)$-connected.

Due to the π_n-isomorphisms just established, we have an exact sequence

$$(\dagger) \qquad \cdots \to \pi_{n+1}\left(L_i'\right) \xrightarrow{g_{i*}'} \pi_{n+1}\left(L_{i-1}'\right) \to \pi_{n+1}\left(g_i'\right) \to 0,$$

for each i. Furthermore, since $n \geq 1$, each g_i' induces a π_1-isomorphism, so we may pass to the universal covers to obtain (by covering space theory and the Hurewicz theorem) isomorphisms:

$$(\ddagger) \qquad \pi_{n+1}\left(g_i'\right) \cong \pi_{n+1}\left(\widetilde{g}_i'\right) \cong H_{n+1}\left(\widetilde{g}_i'\right).$$

Each term in the cellular chain complex $C_*\left(\widetilde{g}_i'\right)$ is a finitely generated $\mathbb{Z}[\pi_1\left(L_i\right)]$-module; so, by Lemma 3.3, $H_{n+1}\left(\widetilde{g}_i'\right)$ is finitely generated.

Applying (‡), we may choose a finite generating set $\{{}^i\bar{\alpha}_j\}_{j=1}^{N_i}$ for each $\pi_{n+1}(g_i')$; and by (†), each ${}^i\bar{\alpha}_j$ may be represented by an ${}^i\alpha_j' \in \pi_{n+1}\left(L_{i-1}'\right)$. By Condition $3'$ we may choose for each ${}^i\alpha_j'$, some ${}^i\beta_j \in \pi_{n+1}(L_i')$ such that $g_{i-1}' \circ g_i'({}^i\beta_j) = g_{i-1}'\left({}^i\alpha_j'\right)$. Let ${}^i\alpha_j = {}^i\alpha_j' - g_i'({}^i\beta_j) \in \pi_{n+1}\left(L_{i-1}'\right)$. Then each ${}^i\alpha_j$ is sent to ${}^i\bar{\alpha}_j$ in $\pi_{n+1}(g_i')$ and $g_{i-1*}'({}^i\alpha_j) = 0 \in \pi_{n+1}\left(L_{i-2}'\right)$. Attach $(n+2)$-cells to each L_{i-1}' to kill the ${}^i\alpha_j$'s. Call the resulting complexes L_i's, and for each i let $k_i : L_i \to L_{i-1}'$ be an extension of g_i'. Then let $g_i : L_i \to L_{i-1}$ be the composition of k_i with the inclusion $L_{i-1}' \hookrightarrow L_{i-1}$. This leads to a diagram like that produced in Step 1, hence the new system $\{L_i, g_i\}$ is pro-equivalent to $\{L_i', g_i'\}$, and thus to $\{K_i, f_i\}$. Moreover, it is easy to check that each g_i is $(n+1)$-connected. Properties 1 and 2 are immediate from the construction and the inductive hypothesis. □

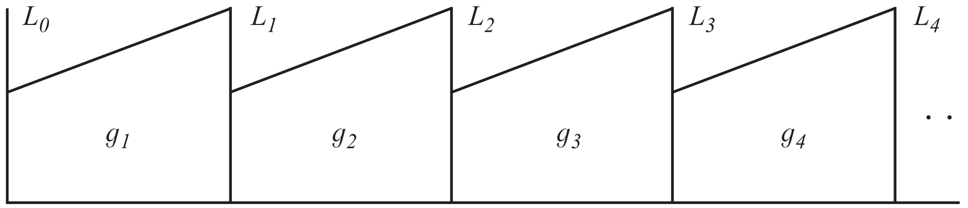
Suppose now that $\{K_i, f_i\}$ has stable pro-π_k for all k. Then, by repeatedly attaching cells to the K_i's, one may obtain pro-equivalent towers with r-connected bonding maps for arbitrarily large r. If $\{K_i, f_i\}$ is finite dimensional it seems reasonable that, once r exceeds the dimension of $\{K_i, f_i\}$, this procedure will terminate with bonding maps that are connected in all dimensions—and thus, homotopy equivalences. Unfortunately, this strategy is too simplistic—in order to obtain r-connected maps we must attach $(r+1)$-cells, thus, the dimensions of the complexes continually exceeds the connectivity of the bonding maps. Roughly speaking, Theorem 3.1 captures the obstruction to making this strategy work.

Proof of Theorem 3.1. Begin with a tower $\{L_i, g_i\}$ of q-dimensional complexes which is pro-equivalent to $\{K_i, f_i\}$ and which has the following properties for all i:

a) g_i is $(q-1)$-connected,
b) for $k \in \{q-2, q-1, q\}$, g_i maps the k-skeleton of L_i into the $(k-1)$-skeleton of L_{i-1},
c) g_{i*} maps $g_{i+1*}(\pi_q(L_{i+1}))$ onto $g_{i*}'(\pi_q(L_i))$, and
d) $g_{i*} : \pi_{q-1}(L_i) \to \pi_{q-1}(L_{i-1})$ is an isomorphism.

A tower satisfying Conditions a) and b) is easily obtainable. Without loss of generality assume that $\{K_i, f_i\}$ is finite dimensional; then apply Lemma 3.4 to $\{K_i, f_i\}$ with $n = \dim\{K_i, f_i\} + 1$, in which case $q = \dim\{K_i, f_i\} + 3$. (**Note.** Although it may seem excessive to allow $\dim\{L_i, g_i\}$ exceed $\dim\{K_i, f_i\}$ by 3, this is done to obtain Condition b) which is key to our argument.) Semistability of pro-π_q gives Condition c)—after passing to a subsequence and relabeling. Then, since pro-π_{q-1} is stable and each $g_{i*} : \pi_{q-1}(L_i) \to \pi_{q-1}(L_{i-1})$ is surjective, we may drop finitely many terms to obtain Condition d).

As in the proof of Lemma 3.4, $\pi_q(g_i)$ and $H_q(\widetilde{g}_i)$ are isomorphic finitely generated $\mathbb{Z}[\pi_1 L_i]$-modules. We will show that, for all i, $H_q(\widetilde{g}_i)$ is projective and that all of these modules are stably equivalent. (This is a pleasant surprise, since the L_i's and g_i's may all be different.) Thus we obtain a single element $[H_q(\widetilde{g}_i)]$ of $\widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{K_i, f_i\})])$. When this element is trivial, i.e., when these modules are stably free, we will show that, by attaching finitely many $(q+1)$-cells to each L_i, all bonding maps can be



made homotopy equivalences. To complete the proof we define $\omega(\{K_i, f_i\})$ to be $(-1)^{q+1}[H_q(\widetilde{g}_i)]$ and show that this element is uniquely determined by $\{K_i, f_i\}$.

Notes. 1) To be more precise, $H_q(\widetilde{g}_i)$ determines an element of $\widetilde{K}_0(\mathbb{Z}[\pi_1(L_i)])$ which may be associated to the appropriate element of $\widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{K_i, f_i\})])$ via the isomorphism induced by projection. We will continue this mild abuse of notation through the remainder of the paper.

2) We have used a factor $(-1)^{q+1}$ (instead of the more concise $(-1)^q$) so that our definition agrees with those already in the literature.

While most of our work takes place in the individual mapping cylinders $M(g_i)$ and their universal covers, there is some important interplay between adjacent cylinders. For this reason, it is useful to view our work as taking place in the "infinite mapping cylinder" shown in Figure 1 (and in its universal cover).

For ease of notation, assume that i has been fixed and consider the pair $\left(M(\widetilde{g}_i), \widetilde{L}_i\right)$ It is a standard fact (see [Co, 3.9]) that $C_*(\widetilde{g}_i)$ is isomorphic to the "algebraic mapping cone" of the chain homomorphism $g_{i*}: C_*\left(\widetilde{L}_i\right) \to C_*\left(\widetilde{L}_{i-1}\right)$. In particular, if the cellular chain complexes $C_*\left(\widetilde{L}_{i-1}\right)$ and $C_*\left(\widetilde{L}_i\right)$ of $\widetilde{L}_{i-1}$ and $\widetilde{L}_i$ are expressed as:

$$(*) \qquad 0 \to D_q \xrightarrow{d_q} D_{q-1} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0, \quad \text{and}$$

$$(**) \qquad 0 \to D'_q \xrightarrow{d'_q} D'_{q-1} \xrightarrow{d'_{q-1}} \cdots \xrightarrow{d'_2} D'_1 \xrightarrow{d'_1} D'_0 \longrightarrow 0,$$

respectively, then $C_*(\widetilde{g}_i)$ is naturally isomorphic to a chain complex

$$0 \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where, for each j,

$$C_j = D'_{j-1} \oplus D_j \quad \text{and} \quad \partial_j(x, y) = (-d'_{j-1}x,\ \widetilde{g}_{i*}x + d_j y).$$

By Condition b), the map $\widetilde{g}_{i*}: D'_j \to D_j$ is trivial for $j \geq q-2$; so, in these dimensions, ∂_j splits as $-d'_{j-1} \oplus d_j$, allowing our chain complex to be written:

$$0 \to \underbrace{D'_q \oplus 0}_{C_{q+1}} \xrightarrow{-d'_q \oplus 0} \underbrace{D'_{q-1} \oplus D_q}_{C_q} \xrightarrow{-d'_{q-1} \oplus d_q} \underbrace{D'_{q-2} \oplus D_{q-1}}_{C_{q-1}} \xrightarrow{-d'_{q-2} \oplus d_{q-1}} \underbrace{D'_{q-3} \oplus D_{q-2}}_{C_{q-2}} \xrightarrow{\partial_{q-2}} \cdots.$$

Since the "minus signs" have no effect on kernels or images of maps, it follows that

$$\ker \partial_{q-1} = \ker(d'_{q-2}) \oplus \ker(d_{q-1}) \tag{3.1}$$

$$\ker \partial_{q} = \ker(d'_{q-1}) \oplus \ker(d_{q}) \tag{3.2}$$

$$\ker \partial_{q+1} = \ker(d'_{q}) \tag{3.3}$$

$$H_{q-1}(\widetilde{g}_i) = \left(\ker(d'_{q-2})\diagup im\left(d'_{q-1}\right)\right) \oplus \left(\ker(d_{q-1})\diagup im\left(d_{q}\right)\right) \tag{3.4}$$

$$H_{q}(\widetilde{g}_i) = \left(\ker(d'_{q-1})\diagup im\left(d'_{q}\right)\right) \oplus \ker(d_{q}) \tag{3.5}$$

$$H_{q+1}(\widetilde{g}_i) = \ker(d'_{q}) \tag{3.6}$$

Since $H_{q-1}(\widetilde{g}_i) = 0$, each summand in Identity 3.4 is trivial. Furthermore, the same reasoning applied to the adjacent mapping cylinder $M(g_{i+1})$ yields an analogous set of identities for $C_*(\widetilde{g}_{i+1})$ in which the "primed terms" become the "unprimed terms". This shows that $\ker(d'_{q-1})\diagup im\left(d'_{q}\right)$ is also trivial. Hence, the first summand in Identity 3.5 is trivial, so $H_q(\widetilde{g}_i) \cong \ker(d_q)$. Identity 3.2 together with Lemma 3.3 then shows that $H_q(\widetilde{g}_i)$ is projective. The same reasoning in $C_*(\widetilde{g}_{i+1})$ shows that $H_q(\widetilde{g}_{i+1}) \cong \ker(d'_q)$ is projective, so by Identity 3.6, $H_{q+1}(\widetilde{g}_i)$ is projective and naturally isomorphic to $H_q(\widetilde{g}_{i+1})$.

Next we show that $H_q(\widetilde{g}_i)$ and $H_{q+1}(\widetilde{g}_i)$ are stably equivalent. Extract the short exact sequence

$$0 \to H_{q+1}(\widetilde{g}_i) \to D'_q \to im\left(d'_q\right) \to 0$$

from above, then recall that $im\left(d'_q\right)$ is equal to $\ker(d'_{q-1})$. The latter is projective, so

$$D'_q \cong H_{q+1}(\widetilde{g}_i) \oplus \ker(d'_{q-1}).$$

Thus $[H_{q+1}(\widetilde{g}_i)] = -\left[\ker(d'_{q-1})\right]$ in $\widetilde{K}_0\left(\check{\pi}_1\left(\{K_i, f_i\}\right)\right)$. But we already know that $[H_q(\widetilde{g}_i)] = [\ker(d_q)]$ and by Identity 3.2, $[\ker(d_q)] = -\left[\ker(d'_{q-1})\right]$. Therefore $[H_q(\widetilde{g}_i)] = [H_{q+1}(\widetilde{g}_i)]$.

To summarize, we have shown that for each i:

- $H_q(\widetilde{g}_i)$ and $H_{q+1}(\widetilde{g}_i)$ are finitely generated and projective,
- $[H_q(\widetilde{g}_i)] = [H_{q+1}(\widetilde{g}_i)]$ in $\widetilde{K}_0\left(\check{\pi}_1\left(\{K_i, f_i\}\right)\right)$, and
- $H_{q+1}(\widetilde{g}_i)$ is naturally isomorphic to $H_q(\widetilde{g}_{i+1})$.

These observations combine to show that each $[H_q(\widetilde{g}_i)]$ determines the same element of $\widetilde{K}_0\left(\check{\pi}_1\left(\{K_i, f_i\}\right)\right)$. Define $\omega\left(\{K_i, f_i\}\right) = (-1)^{q+1}\left[H_q\left(\widetilde{g}_i\right)\right]$.

Claim 1. If $\omega\left(\{K_i, f_i\}\right) = 0$, then $\{K_i, f_i\}$ is stable.

We will show that, by adding finitely many $(q+1)$-cells to each of the above L_i's, we may arrive at a pro-equivalent tower in which all bonding maps are homotopy equivalences.

By assumption, each $\mathbb{Z}\pi_1$-module $H_q(\widetilde{g}_i)$ becomes free upon summation with a finitely generated free module. This may be accomplished geometrically by attaching finitely many q-cells to the corresponding L_{i-1}'s via trivial attaching maps at the basepoints. Each g_{i-1} may then be extended by mapping these q-cells to the basepoint of L_{i-2}. Since this procedure preserves all relevant properties of our tower, we will assume that, for each i, $H_q(\widetilde{g}_i)$ (and therefore $\pi_q(g_i)$) is a finitely generated free $\mathbb{Z}[\pi_1(L_i)]$-module.

Proceed as in Step 2 of the proof of Lemma 3.4 to obtain collections $\{{}^i\alpha_j\}_{j=1}^{N_i} \subset \pi_{n+1}(L_{i-1})$ that correspond to generating sets for the $\pi_q(g_i)$'s and which satisfy $g_{i-1*}({}^i\alpha_j) = 0 \in \pi_q(L_{i-2})$ for all i, j. In addition, we now require that $\{{}^i\alpha_j\}_{j=1}^{N_i}$ corresponds to a free basis for $\pi_q(g_i)$. For each ${}^i\alpha_j$ attach a single $(q+1)$-cell to L_{i-1} to kill that element. Extend each g_i to $g_i' : L_i' \to L_{i-1}'$ as before, thereby obtaining a tower $\{L_i', g_i'\}$ for which all bonding maps are q-connected. Since the $(q+1)$-cells are attached to L_{i-1} along a free basis, we do not create any new $(q+1)$-cycles for the pair $\left(M(\widetilde{g}_i'), \widetilde{L}_i'\right)$, so no new $(q+1)$-dimensional homology is introduced. Moreover, the $(q+1)$-cells attached to L_{i-1} result in $(q+2)$-cells in $M(\widetilde{g}_{i-1}')$ which are attached in precisely the correct manner to kill $H_{q+1}(\widetilde{g}_{i-1})$ without creating any $(q+2)$-dimensional homology—this is due to the natural isomorphism discovered earlier between $H_{q+1}(\widetilde{g}_{i-1})$ and $H_q(\widetilde{g}_i)$. Thus the g_i''s are all $(n+2)$-connected, and since the L_i''s are $(n+1)$-dimensional, this means that the g_i''s are homotopy equivalences.

Claim 2. The obstruction is well defined.

We must show that $\omega(\{K_i, f_i\})$ does not depend on the tower $\{L_i, g_i\}$ chosen at the beginning of the proof. First observe that any subsequence $\left\{L_{k_i}, g_{k_i k_{i-1}}\right\}$ of $\{L_i, g_i\}$ yields the same obstruction. This is immediate in the special case that $\left\{L_{k_i}, g_{k_i k_{i-1}}\right\}$ contains two consecutive terms of $\{L_i, g_i\}$. If not, notice that $\left\{L_{k_i}, g_{k_i k_{i-1}}\right\}$ is a subsequence of $L_{k_1} \leftarrow L_{k_1+1} \leftarrow L_{k_2} \leftarrow L_{k_3} \leftarrow \cdots$, which is a subsequence of $\{L_i, g_i\}$. Hence, the more general observation follows from the special case.

Next suppose that $\{L_i, g_i\}$ and $\{M_i, h_i\}$ are each towers of finite q-dimensional complexes satisfying Conditions a)-d) at the beginning of the proof. Then $\{L_i, g_i\}$ and $\{M_i, h_i\}$ are pro-equivalent; so, after passing to subsequences and relabeling, there exists a homotopy commuting diagram of the form:

$$\begin{tikzcd}
L_0 & & L_1 \arrow[ll, "g_1"'] \arrow[dl, "\mu_1"] & & L_2 \arrow[ll, "g_2"'] \arrow[dl, "\mu_2"] & & L_3 \arrow[ll, "g_3"'] \arrow[dl, "\mu_3"] & & \cdots \arrow[ll, "g_4"'] \\
& M_1 \arrow[ul, "\lambda_1"] & & M_2 \arrow[ll, "h_2"'] \arrow[ul, "\lambda_2"] & & M_3 \arrow[ll, "h_3"'] \arrow[ul, "\lambda_3"] & & \cdots \arrow[ll, "h_4"']
\end{tikzcd}$$

where all λ_i and μ_i are cellular maps. From here we may create a new tower:

$$M_1 \longleftarrow L_2 \longleftarrow M_4 \longleftarrow L_5 \longleftarrow M_7 \longleftarrow L_8 \longleftarrow M_{10} \longleftarrow \cdots$$

where the bonding maps are determined (up to homotopy) by the above diagram. Properties a),c) and d) hold for this tower due to the corresponding properties for $\{L_i, g_i\}$ and $\{M_i, h_i\}$. To see that Property b) holds, note that each bonding map is the composition of a g_i or an h_i with a cellular map. (This is why so many terms were omitted.) Since this new tower contains subsequences which are—up to homotopies of the bonding maps—subsequences of $\{L_i, g_i\}$ and $\{M_i, h_i\}$, our initial observation implies that all determine the same obstruction.

Finally we consider the general situation where $\{L_i, g_i\}$ and $\{M_i, h_i\}$ satisfy Conditions a)-d), but are not necessarily of the same dimension. By the previous case and induction, it will be enough to show that, for a given q-dimensional $\{L_i, g_i\}$, we can find a $(q+1)$-dimensional tower $\{L_i', g_i'\}$ which satisfies the corresponding versions of Conditions a)-d), and which determines the same obstruction as $\{L_i, g_i\}$. In this step, the need for the $(-1)^{q+1}$ factor finally becomes clear.

The tower $\{L_i', g_i'\}$ is obtained by carrying out our usual strategy of attaching a finite collection of $(q+1)$-cells to each L_{i-1} along a generating set for $H_q\left(M(\widetilde{g}_i), \widetilde{L}_i\right)$. The resulting $C_*\left(\widetilde{L}_i'\right)$'s differ from the $C_*\left(\widetilde{L}_i\right)$'s only in dimension $q+1$ where we have introduced finitely generated free modules ${}^iF_{q+1}$. By inserting this term into (*) and rewriting D_q as $im(d_q) \oplus \ker(d_q)$, the chain complex for L_{i-1}' may be written:

$$0 \longrightarrow {}^iF_{q+1} \xrightarrow{d_{q+1}} im(d_q) \oplus \ker(d_q) \xrightarrow{d_q} D_{q-1} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0,$$

By construction, d_{q+1} takes ${}^iF_{q+1}$ onto $\ker(d_q)$ thereby eliminating the q-dimensional homology of the pair $\left(M(\widetilde{g}_i'), \widetilde{L}_i'\right)$. Note however, that we may have introduced new $(q+1)$-dimensional homology. Indeed, by our earlier analysis, $H_{q+1}(\widetilde{g}_i') = \ker(d_{q+1})$. (The original $(q+1)$-dimensional homology of the pair was eliminated—as it was in the unobstructed case—when we attached $(q+1)$-cells to L_i.) By extracting the short exact sequence

$$0 \longrightarrow \ker(d_{q+1}) \longrightarrow {}^iF_{q+1} \longrightarrow \ker(d_q) \longrightarrow 0$$

and recalling that $\ker(d_q) \cong H_q(\widetilde{g}_i)$ is projective, we have

$${}^iF_{q+1} \cong H_{q+1}(\widetilde{g}_i') \oplus H_q(\widetilde{g}_i).$$

So, as elements of $\widetilde{K}_0(\check{\pi}_1(\{K_i, f_i\}))$, $[H_{q+1}(\widetilde{g}_i')] = -[H_q(\widetilde{g}_i)]$. Therefore $(-1)^{q+2}[H_{q+1}(\widetilde{g}_i')] = (-1)^{q+1}[H_q(\widetilde{g}_i)]$, showing that $\{L_i, g_i\}$ and $\{L_i', g_i'\}$ determine the same obstruction. ■

4. Realizing the obstructions

In addition to proving Theorems 3.1 and 3.2, Edwards and Geoghegan showed how to build towers and compacta with non-trivial obstructions. By applying their strategy within our framework, we obtain an easy proof of the following:

Proposition 4.1. *Let G be a finitely presentable group and P a finitely generated projective $\mathbb{Z}[G]$ module. Then there exists a tower of finite 2-complexes $\{K_i, f_i\}$ with stable pro-π_k for all k and $\check{\pi}_1(\{K_i, f_i\}) \cong G$ such that $\omega(\{K_i, f_i\}) = [P] \in \widetilde{K}_0(\mathbb{Z}[G])$.*

By letting $Z = \varprojlim \{K_i, f_i\}$ we immediately obtain:

Proposition 4.2. *Let G be a finitely presentable group and P a finitely generated projective $\mathbb{Z}[G]$ module. Then there exists a compact connected 2-dimensional pointed compactum Z with stable pro-π_k for all k and $\check{\pi}_1(Z) \cong G$ such that $\omega(\{K_i, f_i\}) = [P] \in \widetilde{K}_0(\mathbb{Z}[G])$.*

Proof. Let Q be a finitely generated projective $\mathbb{Z}[G]$ module representing $-[P]$ in $\widetilde{K}_0(\mathbb{Z}[G])$, and so that $F = P \oplus Q$ is finitely generated and free. Let r denote the rank of F. Let K' be a finite pointed 2-complex with $\pi_1(K') \cong G$, then construct K from K' by wedging a bouquet of r 2-spheres to K' at the basepoint. Then $\pi_2(K) \cong H_2\left(\widetilde{K}\right)$ has a summand isomorphic to F which corresponds to the bouquet of 2-spheres. Define a map $f : K \to K$ so that $f\mid_{K'} = id$ and $f_* : \pi_2(K) \to \pi_2(K)$ (or equivalently $\widetilde{f}_* : H_2\left(\widetilde{K}\right) \to H_2\left(\widetilde{K}\right)$) is the projection $P \oplus Q \to P$ when restricted to the F-factor. Note that $H_2\left(\widetilde{f}\right) \cong Q \cong H_3\left(\widetilde{f}\right)$. Obtain the tower $\{K_i, f_i\}$ by letting $K_i = K$ for all $k \geq 0$ and $f_i = f$ for all $k \geq 1$.

To calculate $\omega(\{K_i, f_i\})$ according to the proof of Theorem 3.1, we must attach cells of dimensions $3, 4$ and 5 to each K_i to obtain an equivalent tower $\{L_i, g_i\}$ satisfying Conditions a)-d) of the proof. As we saw in Claim 2 of Theorem 3.1, this procedure simply shifts homology to higher dimensions. In particular, $[H_5(\widetilde{g}_i)] = -[H_2\left(\widetilde{f}\right)] = [P]$, as desired. ☐

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