S^4 admits no uscd into shape S^1's?

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Some history.

Ron Fintushel, building on work of Raymond and Orlik, Montgomery and Yang, classifies (1976-1978) locally smooth circle actions on: homotopy 4-spheres; simply connected 4-manifolds; and then 4-manifolds generally.

Pao (1978) follows with a classification of nonlinear actions.

Plotnick (1982) extends the results of both to homology 4-spheres, and then builds examples of such that admit no effective S^1 action and thus have fundamental groups that cannot belong to a 3-manifold.

Theorem. [Fintushel] Let M^* be the orbit space of a locally smooth S^1 action on the simply connected 4-manifold M, with exceptional orbits E and fixed point set F. Then:

- M^* is a simply-connected 3-manifold with $\partial M^* \subset F^*$.
- The set $F^* - \partial M^*$ is finite, and F^* is nonempty.
- The closure of E^* is a collection of polyhedral arcs and simple closed curves in M^*. The components of E^* are open arcs on which orbit types are constant, and these arcs have closures with distinct endpoints in $F^* - \partial M^*$. (continued on next slide)
- If, in addition, M is a homotopy 4-sphere, then: F is either S^2 or S^0 (in the former case, $E = \emptyset$ and M^* is a homotopy 3-cell with boundary F^*; in the latter case, M^* is a homotopy 3-sphere). In the latter case, if there is only one type of exceptional orbit, E^* is an arc and F^* its endpoints; if there are two types of exceptional orbits, then $E^* \cup F^*$ is a scc separated by F^* into two arcs, on each of which the orbit type is constant.

Conjecture. There is no proper, closed map defined on S^4 such that each of its point preimages is a [shape] circle.

Theorem. *Suppose* $\pi : S^4 \to B$ *is a proper, closed surjection such that* $\tilde{b} = \pi^{-1}(b)$ *has the shape of* S^1*, for all* $b \in B$*;* $dim B < \infty$*; and one more hypothesis stated later in this talk. Then* B *is not a generalized manifold (over* $\mathbb{Q}$*).*

Assuming the existence of the map, we begin a catalogue of facts regarding B:

- $\dim B = 3$;
- B is simply connected (π is a π_1-epimorphism);
- and LC^1 (Dydak).

The Leray sheaves of π.

We let $\mathcal{H}^j = \mathcal{H}^j[\pi] = \mathcal{H}^j[\pi; \mathbb{Q}]$ denote the Leray sheaf in dimension j, where $j = 0, 1$

For each $b \in B$ and $j = 0, 1$, the stalk $\mathcal{H}^j_b$ is isomorphic to $\check{H}^j(\tilde{b}; \mathbb{Q}) \cong H^j(S^1; \mathbb{Q}) \cong \mathbb{Q}$.

The topology on $\mathcal{H}^j$ is discrete when restricted to any stalk.

Aside: a crash course in sheaf topology.

For $b \in B$, let $\widetilde{U}$ be a *saturated* nbhd of $\tilde{b}$. Then there is a saturated nbhd $\widetilde{V} \subseteq \widetilde{U}$ that shape deformation retracts to $\tilde{b}$ in $\widetilde{U}$. For any $b_1 \in V$, there is a map $H^j(\tilde{b}; \mathbb{Q}) \to H^j(\widetilde{b_1}; \mathbb{Q})$, the *j-winding function* of b_1 about b. Note that this function is either an isomorphism or the zero map. Given a section σ of $\mathcal{H}^j$ at b defined on V, the section evaluated at b_1 will naturally correspond to the value of the j-winding function of b_1 around b evaluated at $\sigma(b)$. This defines the topology on $\mathcal{H}^j$.

Clearly, then $\mathcal{H}^0$ is sheaf isomorphic to the constant sheaf $\mathbb{Q} \times B$.

More items for the catalogue ...

Theorem. [Dydak and Walsh] There is an open, dense subset C (the *continuity set*) of B on which $\mathcal{H}^1$ is locally constant.

Definition. Let $K = B - C$, the *degeneracy set.*

Corollary. Then K is nowhere dense in B.

Theorem. [Daverman and Snyder; Snyder] C is a generalized 3-manifold, *i.e.* C is an ANR with local (co)homology of a manifold:

$$H^i(B, B - b; \mathbb{Q}) \cong \mathbb{Q}$$

for $i = 0, 3$ and is trivial for all other i.

Theorem. [Walsh] Via a pseudo-isotopy, we may assume that π is also an open map and, hence, that $\widetilde{K}$ is nowhere dense in S^4.

Theorem. [Shaw] K does not locally separate B and $\dim K \leq 1$.

Aside: the Leray Spectral Sequence.

$H^p(A;\mathcal{H}^q[\pi|_A]) \Rightarrow H^{p+q}(\widetilde{A})$

Since our Leray spectral sequence is lacunary, applied to $\pi|_A$ for any $A \subset B$ (and closed supports) we get:

$$\cdots \to H^i(A) \to H^i(\widetilde{A}) \to H^{i-1}(A;\mathcal{H}^1[\pi|_A]) \to \cdots$$

There is also a relative version, for compact A contained in a subset U of B:

$$\cdots \to H^i(U, U-A) \to H^i(\widetilde{U}, \widetilde{U}-\widetilde{A}) \to H^{i-1}(U, U-A;\mathcal{H}^1[\pi|_A]) \to \cdots$$

Aside: the Fary Spectral Sequence.

Let $B = B_0 \supset B_1 \supset B_2 \supset \cdots$ be a filtration of B by *closed* subsets of B. Let $A_t = B_t - B_{t-1}$. Then

$$\bigoplus_t H^{p+t}_{\Phi|_{A_t}}(A_t;\mathcal{H}^{q-t}[\pi]|A_t) \Rightarrow H^{p+q}(S^4)$$

We note here that $\mathcal{H}^{q-t}[\pi]|A_t$, (in our context) when restricted to A_t is the Leray sheaf $\mathcal{H}^{q-t}[\pi|\widetilde{A_t}]$.

We apply this spectral sequence here using $B_1 = K$ and $B_p = 0$ for $p > 1$. Note then that $C = A_0$ and $K = A_1$.

Continuing ...

Proposition. *$\mathcal{H}^1|_C$ is isomorphic to the constant sheaf $\mathbb{Q}\times C$.* [Proof snapshot: over C, π corresponds to a rational circle bundle over C.]

Proposition. *The sheaf $\mathcal{H}^1$ splits.* We abuse notation and say $\mathcal{H}^1 = \mathcal{H}^1|_C \oplus \mathcal{H}^1|_K$.

Let $A = B$, so $\widetilde{B} = S^4$, and apply the exact sequence (absolute version) from the Leray spectral sequence to get the following for our catalogue:

- $H^1(B)$ is trivial (trivially, since $\pi_1(B)$ is trivial)
- $H^2(B) \cong H^0(B;\mathcal{H}^1) \cong H^0(B;\mathcal{H}^1|_C) \oplus H^0(B;\mathcal{H}^1|_K)$
- $0 \cong H^2(B;\mathcal{H}^1) \cong H^2(B;\mathcal{H}^1|_C) \oplus H^2(B;\mathcal{H}^1|_K)$
- $H^3(B) \cong H^1(B;\mathcal{H}^1|_C) \oplus H^1(B;\mathcal{H}^1|_K)$
- $H^2(B;\mathcal{H}^1) \cong \mathbb{Q}$

NB: If coefficients are not shown, they are $\mathbb{Q}$. Supports for the cohomology are taken to be Ψ, the closed subsets of B. Note, for later, that the support $\Psi|_C$ is the collection of compact subsets of C.

Proposition. *$K \neq \emptyset$.*

Proof sketch: If $K = \emptyset$, then $C = B$ implies that B is a compact generalized (co)homology 3-sphere. Thus, $H^2(B)$ is trivial, which, by the previous list, implies $H^0(B;\mathcal{H}^1)$, the group of global sections, consists of only the zero section. But, as noted before, $\mathcal{H}^1|_{B=C}$ is the constant sheaf.

Since $K \neq \emptyset$, it has an open, dense (non-empty) subset K_1 on which $\mathcal{H}^1|_{K_1}$ is locally constant.

Lemma. *K_1, and hence K, is 1-dimensional.*

Proof sketch: Suppose K_1 is 0-dimensional at $b \in K_1$. Let V be an open set in B, with $b \in V$ such that $\mathcal{H}^1|_{V\cap K_1}$ is constant. Find a nbhd W of b contained in V such that $W \cap K_1 = \emptyset$. Then W admits a section that extends to B. Impossible!

Corollary. *K has no 'totally degenerate' points (and, so, no isolated points).*

We say b is totally degenerate if its 1-winding funtion is identically 0 on its punctured nbhd $V - \{b\}$.

Let $K_2 = K - K_1$ (the *second degeneracy set*), which is nowhere dense in K.

We will add as a simplifying assumption that $K_2 = \emptyset$, *i.e.* $\mathcal{H}^1_K$ is locally constant.

Notice that $\mathcal{H}^1_K$ can*not* be constant, for otherwise a global section on B exists.

Now, we move to add information from the Fary spectral sequence ...

What our Fary spectral sequence looks like in the E_2 term (similar to an S^1-bundle with singularities' - but with complicating differences):

$$\begin{array}{cccc} H^1(K;\mathcal{H}^1|K) & & & \\ H^0_c(C;\mathcal{H}^1|_C) \oplus H^1(K) & H^1_c(C;\mathcal{H}^1|_C) & H^2(C;\mathcal{H}^1|_C) & H^3_c(C;\mathcal{H}^1|_C) \\ H^0_c(C) & H^1_c(C) & H^2(C) & H^3_c(C) \end{array}$$

Using this, and the relative cohomology sequence of the pair (B, K) (with coefficients $\mathbb{Q}$), we can deduce that K is connected. Moreover, $H^1(K) \cong 0$ (from the FSS) and $H^1(K;\mathcal{H}^1|_K) \cong \mathbb{Q}$ (from the LSS). (This latter fact tells us then that $H^3_c(C) \cong \mathbb{Q}$).

Having established these facts, we move to looking at the relative version of the LSS, and leverage the fact that B is assumed to be a 3-gm.

We are then able to prove (this still has the flavor of transformation group theory):

Lemma. *K is a homology 1-manifold.*

But for $n \leq 2$, a homology n-manifold is an n-manifold. Thus $K \cong S^1$.

This last statement is clearly impossible, since $H^1(K) \neq H^1(S^1)$

Question: *Is* there an example of such a map where its image is *not* a generalized manifold?