A STABILIZATION THEOREM FOR OPEN MANIFOLDS

CRAIG R. GUILBAULT

Abstract. In this note we present a characterization of those one-ended open n-manifolds ($n \geq 5$), whose products with the real line are homeomorphic to interiors of compact $(n+1)$-manifolds with boundary.

1. Introduction

This work was motivated by a question asked to me recently by Igor Belegradek.

Question (Belegradek)**.** *Let M^n be an open manifold homotopy equivalent to an embedded compact submanifold, say a torus. Is $M^n \times \mathbb{R}$ homeomorphic to the interior of a compact manifold?*

For the purposes of this talk, we will focus on one-ended, high-dimensional manifolds; in particular, we assume that $n \geq 5$. (Although much of what we will do is valid in all dimensions; and all of what we do can be done without restriction on the number of ends.) We begin with a few standard definitions and examples.

- A manifold M^n is *open* if it is noncompact and has no boundary.
- A subset V of M^n is a *neighborhood of infinity* if $\overline{M^n - V}$ is compact.
- A neighborhood of infinity is *clean* if it is a codimension 0 submanifold and has bicollared boundary in M^n.
- M^n is *one-ended* if each neighborhood of infinity contains a connected neighborhood of infinity. (We assume this for convenience.)

Example 1. $\mathbb{R}^n$ *is an open n-manifold for all $n \geq 1$. If $n \geq 2$, then $\mathbb{R}^n$ is one-ended.*

Example 2. *Let P^n be a compact manifold with non-empty connected boundary. Then int (P^n) is a one-ended open manifold.*

Example 3. *(Disk with infinitely many handles) Let M^2 be the 2-manifold obtained by attaching a countably infinite discrete collection of handles to an open 2-disk.*

Example 4. *(The Whitehead manifold) In* [Wh]*, J.H.C. Whitehead constructed a, now-famous, example of a contractible (thus one-ended) open 3-manifold that is not homeomorphic to $\mathbb{R}^3$.*

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The following observations about the above examples help to motivate our work.

Facts.

a) Clearly Examples 1 and 2 are themselves interiors of compact manifolds; hence, so are their products with $\mathbb{R}$.

b) The manifold M^2 from Exercise 3 is not the interior of a compact 2-manifold; nor is $M^2 \times \mathbb{R}$ the interior of a compact 3-manifold. (*Exercise.* Why?)

c) The Whitehead manifold W^3 is not the interior of any compact 3-manifold; however, it is well-known that $W^3 \times \mathbb{R} \approx \mathbb{R}^4 \approx int\,(B^4)$. In fact, a result of Stallings [St] ensures that the product of *any* contractible n-manifold with a line is homeomorphic to $\mathbb{R}^{n+1}$

Reflection upon the above examples, together with past experience with non-compact manifolds, causes us to generalize our question to:

Generalized Belegradek Question (GBQ). If M^n is open and homotopy equivalent to a finite complex, is $M^n \times \mathbb{R}$ the interior of a compact $(n+1)$-manifold with boundary? (*As noted earlier, we restrict our attention to the case where M^n is one-ended and $n \geq 5$.*)

2. Results

In this section, we outline our solution to the GBQ in the one-ended case. As might be expected of any work on recognizing interiors of compact high-dimensional manifolds, we will employ the following celebrated result:

Theorem 2.1. *(Siebenmann, 1965) A one ended open n-manifold M^n $(n \geq 6)$ is the interior of a compact manifold with boundary iff:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is stable at infinity, and*
(3) *$\sigma_\infty(M^n) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$ is trivial.*

- Here *inward tame* means that for any neighborhood V of infinity, there exists a homotopy $H : V \times [0,1] \to V$ such that $H_0 = id$ and $\overline{H_1(V)}$ is compact. (Equivalently, we may require that all clean neighborhoods of infinity are are finitely dominated.)
- Combined, conditions 1) and 3) are equivalent to requiring that all clean neighborhoods of infinity have finite homotopy type. (For the purposes of this talk, we will refer to this property as *super-tame at infinity.*)

The following straightforward proposition begins our attack on the GBQ.

Proposition 2.2. *Let M^n be a connected open n-manifold.*

(1) *$M^n \times \mathbb{R}$ is inward tame at ∞ iff M^n is finitely dominated.*
(2) *$M^n \times \mathbb{R}$ is super-tame at ∞ iff M^n has finite homotopy type.*

Key Ingredient of Proof. $M^n \times \mathbb{R}$ has arbitrarily small neighborhoods of infinity of the form

$$U = (V \times \mathbb{R}) \cup (M \times [(-\infty, -r] \cup [r, \infty)])$$

where V is a clean neighborhood of infinity in M^n. □

Equipped with Proposition 2.2 and Siebenmann's Theorem, it becomes clear that the answer to the GBQ depends only uponon the π_1-stability at infinity (or the lack thereof) in $M^n \times \mathbb{R}$. Siebenmann must have recognized this back in 1965 when he gave a positive answer to a weaker version of the GBQ—in particular, he allowed himself to cross with $\mathbb{R}^2$ instead of $\mathbb{R}$. The point there was that, by crossing with $\mathbb{R}^2$, π_1-stability at infinity becomes easy. (Verification of this fact is a good exercise.) Before proceeding, we review the meaning of π_1-stability at infinity.

A one-ended open manifold X of dimension at least 5, is π_1 *stable at infinity* if and only if there exists a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of clean neighborhoods of infinity with, $\bigcap V_i = \emptyset$, such that each of the inclusion induced homomorphisms in the corresponding inverse sequence

$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots .$$

are isomorphisms. (Actually, the *definition* of π_1 stable at infinity simply requires that the above inverse sequence be 'pro-stable'. In dimensions ≥ 5, the desired isomorphisms can then be *arranged* using handle trading techniques developed by Siebenmann.)

A positive solution to the GBQ for $n \geq 5$ is obtained by proving the following:

Proposition 2.3. *If M^n is one-ended, open and finitely dominated, then $M^n \times \mathbb{R}$ is π_1-stable at ∞.*

Sketch of Proof. Let

$$U = (V \times \mathbb{R}) \cup (M \times [(-\infty, -r] \cup [r, \infty)])$$

where V is a connected neighborhood of ∞ in M^n.

If $G = \pi_1(M^n)$, then

$$\pi_1(U) \cong G *_H G$$

(a *free product with amalgamation*), where

$$H = image\,(\pi_1(V) \to \pi_1(M^n))$$

So π_1 'at infinity' looks like:

$$(G *_{H_1} G) \twoheadleftarrow (G *_{H_2} G) \twoheadleftarrow (G *_{H_3} G) \twoheadleftarrow \cdots$$

where $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ is a sequence of neighborhoods of ∞ in M^n, and for each i

$$H_i = image\,(\pi_1(V_i) \to \pi_1(M^n)) .$$

To complete the proof, it suffices to show that $H_i = H_j$ for all i, j (when the V_i's are appropriately chosen). This is accomplished by proving:

Claim. *Let K be a compactum into which M^n deforms and let $V' \subseteq V \subseteq M^n - K$ be clean connected neighborhoods of ∞. Then any loop τ in V can be pushed into V' (with base point traveling along a given fixed base ray).*

To prove the claim, begin with an embedded 'base ray' r in M^n and assume τ is based on r. Choose a homotopy $H : M^n \times [0,1] \to M^n$ that pulls M^n into K and is 'nice' near r. (For example, points of r stay in r under H. See the discussion preceding Proposition 3.2 of [GuTi] for details.) Choose a third clean neighborhood $V'' \subseteq V'$ sufficiently small that $\tau \subseteq M^n - V''$. In addition, arrange that $\partial V''$ is connected and r pierces $\partial V''$ transversely in a single point p. Consider the restricted homotopy $H| : \partial V'' \times [0,1] \to M^n$. Adjust $H|$ so that it is transverse to τ. Then $H|^{-1}(\tau)$ will be a finite collection of circles in $\partial V'' \times [0,1]$. By the niceness of $H|$ near r (again see [GuTi, Prop.3.2]), one of these circles, call it τ', is taken in a degree 1 fashion onto τ by $H|$. Using the product structure, τ' can be pushed into $\partial V'' \times \{0\}$ within $\partial V'' \times [0,1]$. Composing this push with $H|$ pushes τ into $\partial V''$ in M^n, as desired. □

We conclude this note with a precise statement of our main result.

Theorem 2.4. *Let M^n be a one-ended open n-manifold ($n \geq 5$), then $M^n \times \mathbb{R}$ is homeomorphic to the interior of a compact $(n+1)$-manifold with boundary if an only if M^n is homotopy equivalent to a finite complex.*

Note. A complete write-up of this work—including the multi-ended case— is in preparation.

References

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Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: craigg@uwm.edu