A CONSTRUCTION OF RIGID CANTOR SETS IN R^3 WITH SIMPLY CONNECTED COMPLEMENT

DENNIS GARITY, DUŠAN REPOVŠ, AND MATJAŽ ŽELJKO

1. Introduction

This is a summary of a talk given by D. Garity on June 11, 2004 at the 21st annual Workshop in Geometric Topology held at the University of Wisconsin in Milwaukee. The results, with complete proofs, are being prepared for publication elsewhere.

A subset $A \subset \mathbb{R}^n$ is *rigid* if whenever $f\colon R^n \to R^n$ is a homeomorphism with $f(A) = A$ it follows that $f|_A = id_A$. There are known examples in R^3 of wild Cantor sets that are either rigid or have simply connected complement. However, until now, no examples were known having both properties.

The class of wild Cantor sets having of simply connected complement known as Bing-Whitehead Cantor sets seemed to suggest that no such example exists because every one-to-one mapping between two finite subsets of a Bing-Whitehead Cantor set $X \subset R^3$ is extendable to a homeomorphism of R^3 which takes X to X (see [Wr4] for details).

Two Cantor sets X and Y in R^3 are said to be *topologically distinct* or *inequivalent* if there is no homeomorphism of R^3 to itself taking X to Y. In this paper we show that in fact uncountably many inequivalent examples of rigid Cantor sets with simply connected complement exist. The key technique used is that of local genus, introduced in [Ze].

Sher proved in [Sh] that there exist uncountably many inequivalent Cantor sets in R^3. He showed that varying the number of components in the Antoine construction leads to these inequivalent Cantor sets. Shilepsky used this result and constructed a rigid Cantor set in R^3 (see [Sl]). Using slightly different approach Wright constructed a rigid Cantor set in R^3 as well (see [Wr2]) and using the Blankinship construction [Bl] Wright later extended

Date: June 11, 2004.

2000 *Mathematics Subject Classification.* Primary 54E45, 54F65 ; Secondary 57M30, 57N10.

Key words and phrases. Wild Cantor set, rigid set, genus of Cantor set, defining sequence.

The first author was supported in part by N.S.F. grants DMS 0139678 and DMS 0104325. The second and third author were supported in part by M.E.S.S. grant 0101-509. All authors were supported in part by M.E.S.S. grant SLO-US 2002/01.

this result to R^n, $n \geq 4$, (see [Wr3]). All these results rely heavily on the linking of the components of defining sequences for the Cantor sets. This linking yields non simply connected complements of the the Cantor sets, so these constructions cannot be modified to give examples of rigid Cantor sets with simply connected complement.

2. Local genus of points in a Cantor set

The following are some basic facts from [Ze] about the genus of a Cantor set and the local genus of points in a Cantor set.

Let $\mathcal{D}(X)$ be the set of all defining sequences for X. Let M be a handlebody. We denote the genus of M by $g(M)$. For a disjoint union of handlebodies $M = \bigsqcup_{\lambda\in\Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda);\ \lambda \in \Lambda\}$.

Let $(M_i) \in \mathcal{D}(X)$ be a defining sequence for a Cantor set $X \subset R^3$. For any subset $A \subset X$ we denote by M_i^A the union of those components of M_i which intersect A. Define

$$\begin{aligned} g_A(X;(M_i)) &= \sup\{g(M_i^A);\ i \geq 0\} \quad \text{and} \\ g_A(X) &= \inf\{g_A(X;(M_i));\ (M_i) \in \mathcal{D}(X)\}. \end{aligned}$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A*. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$. For $A = X$ we call the number $g_X(X)$ *the genus of the Cantor set X* and denote it by $g(X)$.

3. Main results

Lemma 3.1. *Let $X \subset R^3$ be a Cantor set and $A \subset X$ a countable dense subset such that*

(1) *$g_x(X) \leq 2$ for every $x \in X \setminus A$,*
(2) *$g_a(X) > 2$ for every $a \in A$ and*
(3) *$g_a(X) = g_b(X)$ for $a, b \in A$ if and only if $a = b$.*

Then X is a rigid Cantor set in R^3.

The main theorem, which we will prove after detailing the construction, is the following.

Theorem 3.1. *For each increasing sequence $S = (n_1, n_2, \ldots)$ of integers such that $n_1 > 2$, there exists a wild Cantor set in R^3, $X = C(S)$, and a countable dense set $A = \{a_1, a_2, \ldots\} \subset X$ such that the following assertions hold.*

(1) *$g_x(X) \leq 2$ for every $x \in X \setminus A$,*
(2) *$g_{a_i}(X) = n_i$ for every $a_i \in A$ and*
(3) *$R^3 \setminus X$ is simply connected.*

An immediate consequence of this theorem is the following.

Theorem 3.2. *There exist uncountably many inequivalent rigid wild Cantor sets in R^3 with simply connected complement .*

4. THE CONSTRUCTION

Let us fix an increasing sequence $S = (n_1, n_2, \ldots)$ of integers with $n_1 > 2$. We will construct inductively a defining sequence M_1, M_2, ... for a Cantor set $X = C(S)$.

To begin the construction, let M_1 be a unknotted genus n_1 handlebody.

4.1. **Stage $n+1$ if n is odd.** If n is odd then by inductive hypothesis every component of M_n is a handlebody of genus higher than 2. Let N be a genus r component of M_n.

The manifold N can be viewed as an union of r handlebodies of genus 1, $T_1 \cup \ldots \cup T_r$, identified along some 2-discs in their boundaries as shown in Figure 1.

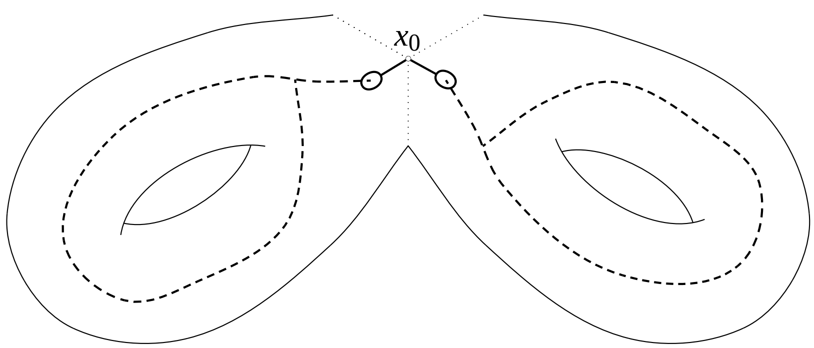

FIGURE 1. Manifold N

We replace the component N of genus r by a single smaller central genus r handlebody and a linked chains of genus 2 handlebodies. We use 6 genus 2 handlebodies for each handle of N. See Figure 2 for the linking pattern in one of the genus 1 handlebodies whose union is N.

Notice that the new components in N are actually unlinked if we regard them as handlebodies in R^3. Stage $n+1$ consists of all the new components constructed as above. The construction can be done so that each new component at stage $n+1$ has diameter less than half of the diameter of the component that contains it at stage n.

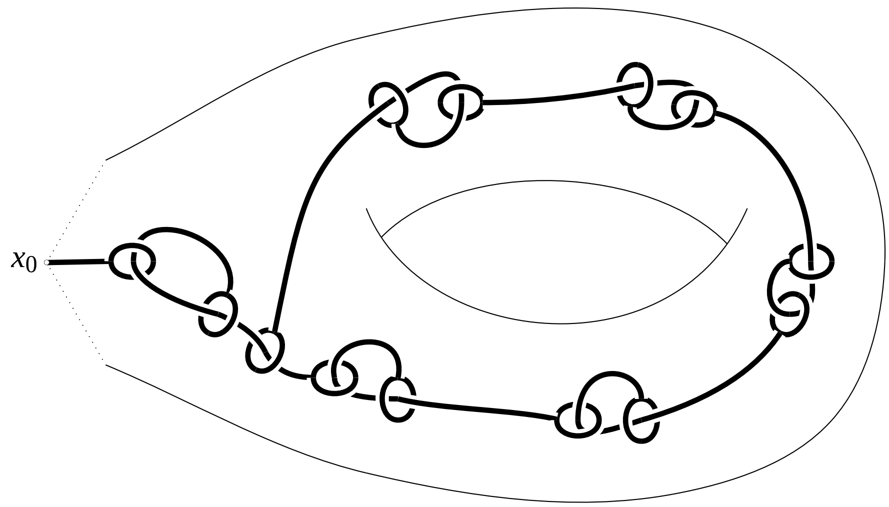

FIGURE 2. Linking along the spine of some handle of N

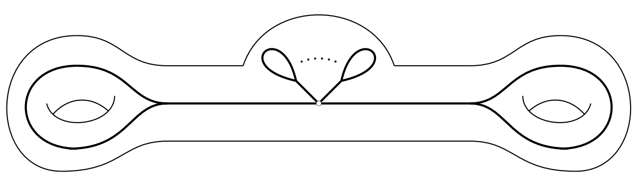

FIGURE 3. Modification in defining sequence

4.2. **Stage $n+1$ if n is even.** If n is even, we replace every genus r torus in M_n, $r > 2$, by a parallel interior copy of itself and every genus 2 torus by an embedded higher genus handlebody as shown in Figure 3.

More precisely, let us assume inductively that there exist handlebodies of genus $n_1, n_2, \dots, n_N$ among the components of M_n. There are also K genus 2 components for some K and we replace one of these genus 2 handlebodies by a genus n_{N+1} handlebody, one by a genus n_{N+2} handlebody, $\dots$ and one by a genus n_{N+K} handlebody. The components of M_{n+1} then consist of handlebodies of genus $n_1, \dots n_{N+K}$.

This completes the inductive description of the defining sequence. Define the Cantor set associated with the sequence S, $X = C(S)$ to be

$$X = \bigcap_i M_i \; .$$

From the construction it is clear that X is a Cantor set.

4.3. **The countable dense subset A.** Each point p in X can be associated with a nondecreasing sequence of positive integers greater than 2 as follows. At stage $2n-1$, p is in a unique component. Let m_n be the genus of this component. The sequence we are looking for is $m_1, m_2, \ldots$. By construction, each m_{n+1} is either equal to m_n or is greater than m_n. It is greater than m_n precisely when the component of stage $2n$ containing p is a genus 2 torus. Let A be the set of points in X for which the associated sequence is bounded. Then A is countable and each point in A is associated with a sequence that is eventually constant. A is dense because each component of each M_i contains a point of$A$.

4.4. **Remaining Details.** The following results can be shown:

- The local genus at points of A is correct
- The local genus at points of $X \setminus A$ is correct,
- The complement of X is simply connected.

5. QUESTIONS

As stated in the introduction Bing-Whitehead Cantor sets have some strong homogeneity properties and therefore are not rigid.

- Does varying the numbers of consecutive Bing links and Whitehead links yield inequivalent Cantor sets? (This number cannot be arbitrary. See [Wr4] for details.)

The construction above gives a rigid Cantor set such that $g_x(X) \leq 2$ for $x \in X \setminus A$ and $g_{a_i}(X) = n_i$ for $a_i \in A$. Hence $g(X) = \infty$.

Let a positive integer r be given.

- Does there exist a rigid Cantor set X such that $g_x(X) = r$ for every $x \in X$? (For $r = 1$ the answer is affirmative. See [Sl], [Wr2].)
- Does there exist a rigid Cantor set X having simply connected complement such that $g_x(X) = r$ for every $x \in X$?

REFERENCES

[Ar] S. Armentrout, *Decompositions of $\mathbf{E}^3$ with a compact 0-dimensional set of nondegenerate elements*, Trans. Amer. Math. Soc. **123** (1966), 165–177

[Ba] A. Babich, *Scrawny Cantor sets are not definable by tori*, Proc. Amer. Math. Soc. **115** (1992), 829–836.

[Bi] R. H. Bing, *Tame Cantor sets in R^3*, Pacific J. Math. **11** (1961), 435–446.

[Bl] W. A. Blankinship, *Generalization of a construction of Antoine*, Ann. of Math. (2) **53** (1951), 276–297.

[Da] R. J. Daverman, *Decompositions of Manifolds*, Pure and Appl. Math. **124**, Academic Press, Orlando, 1986.

[Ma] J. M. Martin, *A rigid sphere*, Fund. Math. **59** (1966), 117–121.

[Os] R. P. Osborne, *Embedding Cantor sets in a manifold, Part I: Tame Cantor sets in R^n*, Michigan Math. J. **13** (1966), 57–63.

[Sh] R. B. Sher, *Concerning wild Cantor sets in R^3*, Proc. Amer. Math. Soc. **19** (1968), 1195–1200.

[Sl] A. C. Shilepsky, *A rigid Cantor set in $\mathbb{E}^3$*, Bull. Acad. Polon. Sci. **22** (1974), 223–224.

[Sk] R. Skora, *Cantor sets in S^3 with simply connected complements*, Topol. Appl. **24** (1986), 181–188.

[Wr1] D. G. Wright, *Cantor sets in 3-manifolds*, Rocky Mountain J. Math. **9** (1979), 377–383.

[Wr2] D. G. Wright, *Rigid sets in manifolds*, Geometric and Algebraic Topology, Banach Center Publ. **18**, PWN, Warsaw 1986, 161–164.

[Wr3] D. G. Wright, *Rigid Cantor sets in R^n*, Pacific J. Math. **121** (1986), 245–456.

[Wr4] D. G. Wright, *Bing-Whitehead Cantor sets*, Fund. Math. **132** (1989), 105–116.

[Ze] M. Željko, *Rigid Cantor sets*, Rocky Mountain J. Math., to appear.

MATHEMATICS DEPARTMENT, OREGON STATE UNIVERSITY, CORVALLIS, OR 97331, U.S.A.

E-mail address: garity@math.oregonstate.edu

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19, P.O.BOX 2964, LJUBLJANA, SLOVENIA

E-mail address: dusan.repovs@uni-lj.si

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19, P.O.BOX 2964, LJUBLJANA, SLOVENIA

E-mail address: matjaz.zeljko@fmf.uni-lj.si