# PRIME IDEALS AND RADICALS IN RINGS GRADED BY CLIFFORD SEMIGROUPS

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In this paper we continue our study of the ideal structure of the direct sum of a directed system of rings indexed by a semigroup begun in [1], with emphasis on describing the prime ideals and radicals of semigroup rings and semigroup-graded rings. This time we concentrate on semigroups that fail to satisfy condition (†) of our orginal article but have a sufficient quantity of nearly central idempotents, and we reduce the description of the prime ideals and radicals to the case of group rings and prime families over systems of group rings. Our results apply to Clifford semigroups, commutative semigroups for which every element has a power lying in a subgroup, and some more general classes of semigroups.

We extend the results of our previous paper [1], whose notation and approach we retain. In that paper we showed how to describe the prime ideals and various radicals of a special type of semigroup-graded ring that generalizes the notion of a strong supplementary semilattice sum of rings. In the current paper, we extend these results to systems of rings over semigroups having the following properties: (i) all prime ideals are completely prime; (ii) $e\alpha e\beta e = e\alpha\beta e$ for all $\alpha, \beta, e$ with $e$ idempotent; and (iii) for every $\alpha$, there is an idempotent $e$ lying in exactly the same prime ideals as $\alpha$. Examples of such semigroups include Clifford semigroups and commutative semigroups for which every element has some power lying in a subgroup (for example, commutative periodic semigroups). In fact, we show any semigroup having these three properties has a largest Clifford semigroup homomorphic image, and so we first discuss the Clifford semigroup case and then show how to deduce results in the general case from this case. We refer the reader to [1] for more details and for unexplained terminology in what follows, and also for a discussion of related results in the literature, and we refer the reader to [2] and [5] for background on semigroups. We also refer the reader to [4], where A. V. Kelarev has outlined a general approach to describing the structure of radicals of semigroup-graded rings. These general results are not as detailed and concrete as those derived in this paper and in some other specific cases.

Throughout this paper, $\Omega$ will denote a semigroup, $E = E(\Omega)$ will denote its set of idempotents, and $\prec$ will denote the *ideal preorder* on $\Omega$, defined by $\alpha \prec \beta$ if $\beta$ is

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in the ideal $(\alpha)$ of $\Omega$ generated by $\alpha$. Recall that a *system of rings* over $(\Omega, \prec)$ is a collection $(R_\alpha)_{\alpha\in\Omega}$ of rings, together with ring homomorphisms $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$, such that $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ whenever $\alpha \prec \beta \prec \gamma$ and such that $\phi_{\alpha,\alpha} = \mathrm{id}_{R_\alpha}$ for all $\alpha$. Throughout this paper, $\mathcal{R}$ will denote a system of rings over $\Omega$, and $R = \oplus\mathcal{R} = \oplus_{\alpha\in\Omega} R_\alpha$ will denote the direct sum of $\mathcal{R}$, with multiplication defined as follows. We identify each $R_\alpha$ with its image in $R$, and for $r_\alpha \in R_\alpha, r'_\beta \in R_\beta$, we define $r_\alpha \cdot r'_\beta = \phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta)$. This $R$ is an $\Omega$-graded ring. One example of this construction is the semigroup ring $S[\Omega]$, which is the direct sum of a system $\mathcal{R}$ in which each $R_\alpha \cong S$ and each map $\phi_{\alpha,\beta}$ is an isomorphism.

Recall that a proper ideal of a semigroup is said to be *prime* if whenever it contains a product of ideals, it contains one of the factors, and is said to be *completely prime* if whenever it contains a product of elements, it contains one of the factors. We regard $\emptyset$ as a prime and completely prime ideal. In [1] we showed that if $\Omega$ is a semigroup in which all elements are idempotent and which satisfies the identity $xyxzx = xyzx$, then the prime ideals in the system sum $R = \oplus\mathcal{R}$ correspond to the prime ideals of the direct limits (relative to the ideal preorder) $\varinjlim_{\alpha\in\Omega\setminus\Phi} R_\alpha$ for prime ideals $\Phi$ of $\Omega$, and we gave an explicit description of the prime ideals in such a direct limit using the notion of a prime family, a special family of ideals of the rings $R_\alpha$, $\alpha \in \Omega \setminus \Phi$. We also gave an explicit description of the radical of $R$ for any radical that is hereditary and directed (for example, the Jacobson radical or the upper nil radical). In Section 1 of the current paper we assume $\Omega$ is a Clifford semigroup and show how to apply the results above to describe prime ideals and radicals — this is fairly straightforward and involves replacing $\Omega$ by its set of idempotents and the $R_\alpha$ by certain group rings. In Sections 2 and 3 we generalize these results to the case of a semigroup $\Omega$ satisfying conditions (i),(ii),(iii) above. We do this by showing $\Omega$ has a greatest Clifford semigroup homomorphic image and showing that we can lift the results of Section 1. We thus show the prime ideals of $R$ correspond to the prime ideals of direct limits $\varinjlim_{e\in E'\setminus\Phi} R_e[\Gamma_e]$ for prime ideals $\Phi$ of a semilattice $E'$ of idempotents of $\Omega$, where the $\Gamma_e$ are maximal subgroups of $\Omega$, and we give an explicit description of the radical of $R$ for any directed, hereditary radical.

The paper [1] was written with S. S. Stalder and M. L. Teply. I would like to thank Teply for showing me some work he had done toward extending the results of [1]; this inspired me to seek more general results using the philosophy of [1] and led to my writing the current paper.

## 1. System sums over Clifford semigroups

Recall that $\Omega$ is a *Clifford semigroup* if it is a regular semigroup and all of its idempotents are central. One can show $\Omega$ is a Clifford semigroup if and only if $E = E(\Omega)$ is a semilattice and $\Omega$ is the disjoint union of subgroups $\Omega_e$ for $e \in E$, where $\Omega_e$ has identity $e$ and $\Omega_e \cdot \Omega_f \subseteq \Omega_{ef}$. In this case, $\Omega$ is actually a strong semilattice of the subgroups $\Omega_e$. (See [2, §4.2] or [5, §III.7].) The $\Omega_e$ are precisely the maximal subgroups of $\Omega$. In this section we show how to extend the results of [1] to the case of Clifford semigroups.

Assume that $\Omega$ is a Clifford semigroup, and for each $e \in E$, set $R^{(e)} = \oplus_{\alpha\in\Omega_e} R_\alpha$: this is a subring of $R$ and $R = \oplus_{e\in E} R^{(e)}$. We wish to show the $R^{(e)}$ form a system of rings over $E$ whose direct sum is $R$. It is easy to see that for $e, f \in E$, we have $e \prec f$ in either $\Omega$ or in $E$ if and only if $ef = f$. When this is the case, define an additive map $\phi^{(e,f)} : R^{(e)} \to R^{(f)}$ by setting $\phi^{(e,f)}(r_\alpha) = \phi_{\alpha,\alpha f}(r_\alpha)$ for $r_\alpha \in R_\alpha$. This map is a homomorphism of rings, since if $\alpha, \beta \in \Omega_e$ and $r_\alpha \in R_\alpha, r'_\beta \in R_\beta$, we have

$$\begin{aligned}\phi^{(e,f)}(r_\alpha \cdot r'_\beta) &= \phi_{\alpha\beta,\alpha\beta f}(\phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta)) = \phi_{\alpha,\alpha f\beta f}(r_\alpha)\phi_{\beta,\alpha f\beta f}(r'_\beta)\\ &= \phi_{\alpha,\alpha f}(r_\alpha)\cdot\phi_{\beta,\beta f}(r'_\beta) = \phi^{(e,f)}(r_\alpha)\cdot\phi^{(e,f)}(r'_\beta).\end{aligned}$$

Thus the rings $R^{(e)}$ together with the maps $\phi^{(e,f)}$ form a system of rings over $E$. To show that the direct sum of this system is $R$, we need to show the products agree. Let $\alpha \in \Omega_e, \beta \in \Omega_f$ for $e, f \in E$ and let $r_\alpha \in R_\alpha, r'_\beta \in R_\beta$. The product in $R$ is given by $r_\alpha \cdot r'_\beta = \phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta)$, while the product in the direct sum is given by

$$\begin{aligned}r_\alpha \cdot r'_\beta &= \phi^{(e,ef)}(r_\alpha)\phi^{(f,ef)}(r'_\beta) = \phi_{\alpha,\alpha ef}(r_\alpha)\phi_{\beta,\beta ef}(r'_\beta)\\ &= \phi_{\alpha,\alpha ef\beta ef}(r_\alpha)\phi_{\beta,\alpha ef\beta ef}(r'_\beta) = \phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta).\end{aligned}$$

We also note that since $\Omega_e$ is a group, the maps $\phi_{\alpha,e}$ are all isomorphisms for $\alpha \in \Omega_e$, and so the ring $R^{(e)}$ is isomorphic to the group ring $R_e[\Omega_e]$ via the map $\oplus_{\alpha\in\Omega_e} r_\alpha \mapsto \sum_{\alpha\in\Omega_e} \phi_{\alpha,e}(r_\alpha)\alpha$. If we redefine our system maps accordingly, they become $\phi^{(e,f)}(\sum_{\alpha\in\Omega_e} s_\alpha\,\alpha) = \sum_{\alpha\in\Omega_e} \phi_{e,f}(s_\alpha)\,\alpha f$, where each $s_\alpha \in R_e$. We then see that $R$ itself is isomorphic to the direct sum of the system over $E$ with $R^{(e)} = R_e[\Omega_e]$ and maps $\phi^{(e,f)}$ as just defined.

We have thus realized $R$ as the direct sum of a system of rings over a semilattice, and so we may apply the results of [1]. The results leading up to Theorem 2.7 of [1] tell us that prime ideals of $R$ correspond bijectively to prime ideals of the direct limits $\varinjlim_{e\in E\setminus\Phi} R_e[\Omega_e]$ as $\Phi$ ranges over the prime ideals of $E$. (Note that the system maps $\phi^{(e,f)}$ map $R_e$ into $R_f$ and $\Omega_e$ into $\Omega_f$, so we may form separate direct limits and identify $\varinjlim_{e\in E\setminus\Phi} R_e[\Omega_e]$ with the group ring $(\varinjlim_{e\in E\setminus\Phi} R_e)[\varinjlim_{e\in E\setminus\Phi} \Omega_e]$.) Theorem 2.7 of [1] then tells us that these prime ideals correspond to prime families $(\Phi, \mathcal{P})$ as defined there.

Theorem 4.5 of [1] tells us that if $\rho$ is a radical and either $E$ is finite or $\rho$ is directed and hereditary, then

$$\rho(R) = \{\, \oplus_{e\in E} r^e \in \oplus_{e\in E} R_e[\Omega_e] \;\Big|\; \sum_{e\in E \text{ with } ef=f} \phi^{(e,f)}(r^e) \in \rho(R_f[\Omega_f]) \text{ for all } f \in E \,\}.$$

Moreover, $\rho(R) = \oplus_{e\in E} \rho(R_e[\Omega_e])$ if and only if $\phi^{(e,f)}(\rho(R_e[\Omega_e])) \subseteq \rho(R_f[\Omega_f])$ whenever $e, f \in E$ and $ef = f$.

We now state these results formally, using the original form of $R$. To obtain part (3) of the theorem, use the remarks prior to Proposition 4.5 of [1]. Other descriptions of the Jacobson radical of $R$ have been given in [3] and [6].

**Theorem 1.1.** *Let $\Omega$ be a Clifford semigroup with set of idempotents $E$, let $\mathcal{R}$ be a system of rings over $\Omega$, and let $R = \oplus\mathcal{R}$. Then*

(1) *The prime ideals in* $R$ *correspond bijectively to prime families* $(\Phi, \mathcal{P})$ *with* $\Phi$ *a prime ideal of* $E$ *and* $\mathcal{P}$ *a prime family over the group rings* $R_e[\Omega_e]$ *with* $e \in E \setminus \Phi$, *as in Theorem 2.7 of* [1].

*Explicitly, this correspondence sends a prime ideal* $P$ *of* $R$ *to the pair* $(\Phi, \mathcal{P} = (P_e)_{e \in E \setminus \Phi})$ *where* $\Phi = \{ e \in E \mid R_f \subseteq P \text{ for all } f \in E \text{ with } ef = f \}$ *and* $P_e$ *is the ideal of* $R_e[\Omega_e]$ *which corresponds to* $P \cap \oplus_{\alpha \in \Omega_e} R_\alpha$. *The inverse sends a pair* $(\Phi, \mathcal{P})$ *to the set of all elements* $r = \oplus_{\alpha \in \Omega} r_\alpha \in R$ *such that* $\sum_{\alpha \text{ with } f \in (\alpha)} \phi_{\alpha,f}(r_\alpha)\alpha f \in P_f$ *for any* $f \in E \setminus \Phi$ *such that* $f \in (\alpha)$ *for all* $\alpha \in \operatorname{supp} r \setminus \cup_{e \in \Phi} \Omega_e$.

(2) *Let* $\rho$ *be a radical and assume either* $E$ *is finite or* $\rho$ *is directed and hereditary. Then*

$$\rho(R) = \{ \oplus_{\alpha \in \Omega} r_\alpha \mid \sum_{\alpha \text{ with } f \in (\alpha)} \phi_{\alpha,f}(r_\alpha)\alpha f \in \rho(R_f[\Omega_f]) \text{ for all } f \in E \}.$$

*Moreover,* $\rho(R) = \{ \oplus_{\alpha \in \Omega} r_\alpha \mid \oplus_{\alpha \in \Omega_e} \phi_{\alpha,e}(r_\alpha)\alpha \in \rho(R_e[\Omega_e]) \text{ for all } e \in E \}$ *if and only if* $\phi^{(e,f)}(\rho(R_e[\Omega_e])) \subseteq \rho(R_f[\Omega_f])$ *whenever* $e, f \in E$ *and* $ef = f$.

(3) *If* $\rho$ *is an arbitrary radical and* $\rho(R_e[\Omega_e]) = 0$ *for all* $e \in E$, *then* $\rho(R) = 0$. ■

In the case of a semigroup ring $S[\Omega]$, the maps $\phi^{(e,f)}$ are given by $\phi^{(e,f)}(r^e) = r^e f$, and the identification of $S[\Omega]$ with $\oplus_{e \in E} S[\Omega_e]$ is automatic. Thus we obtain the following.

**Corollary 1.2.** *Let* $\Omega$ *be a Clifford semigroup with set of idempotents* $E$, *let* $S$ *be a ring, and let* $S[\Omega]$ *be the semigroup ring. Then*

(1) *The prime ideals of* $S[\Omega]$ *correspond bijectively to prime ideals over the group rings* $S[\varinjlim_{e \in E \setminus \Phi} \Omega_e]$ *as* $\Phi$ *ranges over the prime ideals of* $E$.

*Thus prime ideals of* $S[\Omega]$ *correspond bijectively to prime families* $(\Phi, \mathcal{P})$ *with* $\Phi$ *a prime ideal of* $E$ *and* $\mathcal{P}$ *a prime family over the group rings* $S[\Omega_e]$ *with* $e \in E \setminus \Phi$, *as in Theorem 1.1*(1).

(2) *Let* $\rho$ *be a radical and assume either* $E$ *is finite or* $\rho$ *is directed and hereditary. Then*

$$\rho(S[\Omega]) = \{ \oplus_{e \in E} r^e \in \oplus_{e \in E} S[\Omega_e] \mid \sum_{e \in E \text{ with } ef = f} r^e f \in \rho(S[\Omega_f]) \text{ for all } f \in E \}.$$

*Moreover,* $\rho(S[\Omega]) = \oplus_{e \in E} \rho(S[\Omega_e])$ *if and only if* $\rho(S[\Omega_e])f \subseteq \rho(S[\Omega_f])$ *whenever* $e, f \in E$ *with* $ef = f$.

(3) *If* $\rho$ *is a radical and* $\rho(S[G]) = 0$ *for all maximal subgroups* $G$ *of* $\Omega$, *then* $\rho(S[\Omega]) = 0$. ■

## 2. Some special semigroups and congruences

For the rest of this paper, we assume $\Omega$ is a semigroup having the following three properties:

(i) All prime ideals of $\Omega$ are completely prime.

(ii) If $\alpha, \beta \in \Omega, e \in E$, then $e\alpha e\beta e = e\alpha\beta e$.

(iii) For every $\alpha \in \Omega$, there is an $e \in E$ such that $\alpha$ and $e$ are elements of exactly the same prime ideals of $\Omega$.

In this section we will derive some properties of such a semigroup $\Omega$ and we will in particular give a description of the least Clifford semigroup congruence on $\Omega$.

We denote the least semilattice congruence on $\Omega$ by $\eta$: recall that when property (i) holds, we have $\alpha\,\eta\,\beta$ if and only if $\alpha, \beta$ belong to exactly the same prime ideals of $\Omega$. (See [5, §II.2].) Thus (iii) could be re-stated as: for every $\alpha \in \Omega$, there is an $e \in E$ with $\alpha\,\eta\,e$. For each $e \in E$, let $\Omega_e$ denote the $\eta$-class of $e$ and let $\Gamma_e = e\Omega_e e$. Both $\Gamma_e$ and $\Omega_e$ are subsemigroups of $\Omega$, and we will show that $\Gamma_e$ is a group.

Obviously any commutative semigroup satisfies (i) and (ii); it satisfies (iii) if and only if every archimedean component contains an idempotent. This is equivalent to the assumption that some power of every element lies in a subgroup. (See [2, §4.3].) Thus any commutative periodic semigroup satisfies (iii), since in this case, every element has a power that is idempotent. A Clifford semigroup satisfies these three conditions, as does any regular band (that is, any band satisfying the identity (ii)).

Recall Green's relation $\mathfrak{J}$: we define $\alpha\,\mathfrak{J}\,\beta$ if $(\alpha) = (\beta)$. This is an equivalence relation, but not generally a congruence. We will introduce one more relation, but first we will list some properties of semigroups satisfying (i), (ii), and (iii).

**Lemma 2.1.** *Let $\alpha \in \Omega, e \in E$. If $\alpha\,\eta\,e$, then $e \in (\alpha)$.*

*Proof.* Suppose $e \notin (\alpha)$ and let $I$ be the union of all ideals of $\Omega$ containing $\alpha$ but not $e$. Then a standard argument shows that $I$ is prime; this contradicts the fact that $\alpha\,\eta\,e$. ∎

**Lemma 2.2.** *Let $\alpha, \beta \in \Omega$, $e, f \in E$.*

1. *If $\alpha, \beta \in (e)$, then $\alpha e \beta = \alpha\beta$.*
2. *$e \prec f$ if and only if $fef = f$.*
3. *$efe, efef, fef, fefe$ are all idempotents and they are all $\mathfrak{J}$-equivalent.*

*Proof.* (1) We have $\alpha = \gamma e \gamma'$, $\beta = \delta e \delta'$ for some $\gamma, \gamma', \delta, \delta' \in \Omega$. By property (ii),

$$\alpha e \beta = \gamma(e\gamma' e \delta e)\delta' = \gamma e \gamma' \delta e \delta' = \alpha\beta.$$

(2) If $f \in (e)$, we have $fef = ff = f$ by (1). If $fef = f$, certainly $f \in (e)$, so $e \prec f$.

(3) Idempotence is straightforward; for example $efeefe = efefe = effe = efe$. To see that these elements are $\mathfrak{J}$-equivalent, note that certainly $(efef) \subseteq (efe)$. But $efef \cdot e = efe$ as above, so $(efef) = (efe)$. Similarly, $efe = e \cdot fef \cdot e$ and $fef = f \cdot efe \cdot f$, so $(efe) = (fef)$. ∎

**Corollary 2.3.** *If $e, f \in E$, then the following are equivalent.*

1. *$e\,\eta\,f$.*
2. *$e\,\mathfrak{J}\,f$.*
3. *$efe = e$ and $fef = f$.*

*Proof.* (2) $\Leftrightarrow$ (3) follows from Lemma 2.2(2), and clearly (2) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 2.1. ∎

The next lemma will be used when we define our new congruence, and when we study prime ideals in Section 3.

**Lemma 2.4.** *Suppose* $\alpha, \beta \in \Omega$, $e \in E$. *If* $e\alpha e = e\beta e$, *then* $\gamma\alpha\delta = \gamma\beta\delta$ *for all* $\gamma, \delta \in (e)$.

*Proof.* By applying Lemma 2.2(1) repeatedly, we see that

$$\gamma\alpha\delta = \gamma e\alpha e\delta = \gamma e\beta e\delta = \gamma\beta\delta.$$

■

**Proposition 2.5.** *For each* $e \in E$, *the largest subgroup of* $\Omega$ *with identity* $e$ *is* $\Gamma_e = \{ e\alpha e \mid \alpha \,\eta\, e \} = \{ \alpha \mid \alpha \,\eta\, e$ *and* $e\alpha e = \alpha \}$.

*Proof.* Since $e$ is an idempotent, it is trivially an identity for $\Gamma_e$. Let $\alpha \in \Gamma_e$: by Lemma 2.1, we have $e \in (\alpha)$, say $e = \beta\alpha\gamma$ for some $\beta, \gamma \in \Omega$. Multiplying by $e$ and applying property (ii), we obtain

$$e = e\beta\alpha\gamma e = e\beta e\alpha e\gamma e.$$

If we replace $\beta$ by $e\beta e$ and $\gamma$ by $e\gamma e$, we may assume $\beta, \gamma \in (e)$. Plainly $e \in (\beta)$ and $e \in (\gamma)$, so we have $(\beta) = (e) = (\gamma)$, and this implies $\beta \,\eta\, e \,\eta\, \gamma$. Thus $\alpha, \beta, \gamma$ all lie in $\Gamma_e$.

Now suppose $f$ is an idempotent in $\Gamma_e$. Since $e \,\eta\, f$, we have $e = efe = f$ (because $e$ is the identity in $\Gamma_e$). Thus $e$ is the only idempotent in $\Gamma_e$. But note $\alpha\gamma(\beta\alpha\gamma)\beta = \alpha\gamma e\beta = \alpha\gamma\beta$, so $\alpha\gamma\beta$ is idempotent. This implies $\alpha\gamma\beta = e$. A similar calculation shows $\gamma\beta\alpha = e$, so $\alpha$ has inverse $\gamma\beta$ in $\Gamma_e$.

Suppose $G$ is a subgroup of $\Omega$ containing $e$ and let $\alpha \in G$. Then $\alpha = e\alpha \in (e)$ and there exists $\beta \in G$ with $e = \alpha\beta \in (\alpha)$. This shows $(e) = (\alpha)$, so $e \,\eta\, \alpha$. Thus $\alpha = e\alpha e \in \Gamma_e$; this proves $G \subseteq \Gamma_e$. ■

We now define a relation $\equiv$ on $\Omega$. We define $\alpha \equiv \beta$ if there is an $e \in E$ with $\alpha \,\eta\, e \,\eta\, \beta$ and $e\alpha e = e\beta e$. Lemma 2.4 and Corollary 2.3 tell us that then $f\alpha f = f\beta f$ for any $f \in E$ with $f \,\eta\, e$, so this definition is independent of the choice of $e$. This immediately implies $\equiv$ is an equivalence relation. To see that it is a congruence, suppose $\alpha \equiv \beta$ and $e \in E$ is as above, and let $\gamma \in \Omega$. By property (iii), there is an $f \in E$ with $\gamma \,\eta\, f$. We know from Lemma 2.2(3) that $efe \in E$, and clearly $\beta\gamma \;\eta\; \alpha\gamma \;\eta\; ef \;\eta\; efe$. Now

$$ef(e\alpha\gamma e)fe = efe\alpha e\gamma efe = ef(e\beta e\gamma e)fe = efe\beta\gamma efe,$$

so $\alpha\gamma \equiv \beta\gamma$. Similarly, $\gamma\alpha \equiv \gamma\beta$. This proves $\equiv$ is a congruence. Note that if $e, f \in E$, then $e \equiv f$ if and only if $e \,\eta\, f$ (by Corollary 2.3).

If $\Omega$ is a Clifford semigroup, then the congruence $\equiv$ is just equality. To see this, note that the $\eta$-classes are just the groups $\Omega_e$, and $e$ is an identity in $\Omega_e$, so the relation $\equiv$ is certainly trivial in each of these classes. For general semigroups having properties (i),(ii),(iii), every element has a unique $\equiv$-representative in the subgroup $\Gamma_e$, for any $e$ in its $\eta$-class.

**Theorem 2.6.** *The congruence* $\equiv$ *is the least Clifford semigroup congruence on* $\Omega$. *If* $E'$ *is a subset of* $E$ *containing exactly one element from each* $\eta$*-class in* $E$, *then the natural map* $\pi : \Omega \to \overline{\Omega} = \Omega/\equiv$ *is a bijection onto* $\overline{\Omega}$ *when restricted to* $\cup_{e \in E'} \Gamma_e$, *and all the idempotents in* $\overline{\Omega}$ *have the form* $\pi(e)$ *for* $e \in E'$.

*Proof.* If $\alpha \in \Omega$, then $\alpha \equiv e\alpha e$ for the unique $e \in E'$ with $e \,\eta\, \alpha$, so $\pi(\alpha) = \pi(e\alpha e)$. Thus $\overline{\Omega} = \pi(\cup_{e\in E'}\Gamma_e)$. If $\alpha, \beta \in \cup_{e\in E'}\Gamma_e$ and $\pi(\alpha) = \pi(\beta)$, then $\alpha \equiv \beta$ and so there is a unique $e \in E'$ with $\alpha \,\eta\, e \,\eta\, \beta$ and $\alpha = e\alpha e = e\beta e = \beta$. This proves $\pi$ restricted to $\cup_{e\in E'}\Gamma_e$ is a bijection.

Thus $\overline{\Omega}$ is the disjoint union of the $\pi(\Gamma_e)$ for $e \in E'$. Since each $\Gamma_e$ is a group by Proposition 2.5, so is each $\pi(\Gamma_e)$. This proves that $\overline{\Omega}$ is regular. In addition, any idempotent in $\overline{\Omega}$ must be an identity of one of the $\pi(\Gamma_e)$ and so must be one of the $\pi(e)$.

Suppose $\alpha \in \Omega, e, f \in E$ and $\alpha \,\eta\, e$. We wish to show $\alpha f \equiv f\alpha$. By Lemma 2.2(3), $\alpha f \,\eta\, ef \,\eta\, efe$, and

$$e(fe\alpha f)efe = efe(f\alpha fef)e = efef\alpha efe.$$

This proves $\alpha f \equiv f\alpha$ and shows that idempotents are central in $\overline{\Omega}$, and so proves $\overline{\Omega}$ is a Clifford semigroup.

Now let $\sim$ be a congruence on $\Omega$ such that $\Omega/\sim$ is a Clifford semigroup, and let $\theta : \Omega \to \Omega/\sim$ be the natural map. Suppose $\alpha, \beta \in \Omega$ satisfy $\alpha \equiv \beta$, so there is an $e \in E$ with $\alpha \,\eta\, e \,\eta\, \beta$ and $e\alpha e = e\beta e$. Applying $\theta$, we see that $\theta(e)$ is idempotent in $\Omega/\sim$. Moreover, the relation $\theta(\alpha) \,\eta\, \theta(e) \,\eta\, \theta(\beta)$ still holds — this is valid for any homomorphism. In addition $\theta(e)\theta(\alpha)\theta(e) = \theta(e)\theta(\beta)\theta(e)$ holds. This implies $\theta(\alpha) = \theta(\beta)$, since $\theta(e)$ is an identity in its $\eta$-class (as $\Omega/\sim$ is a Clifford semigroup). Thus $\alpha \sim \beta$. ∎

## 3. System sums again

We continue to assume that the semigroup $\Omega$ has the properties (i),(ii),(iii) listed at the start of Section 2, and we let $\equiv$ denote the equivalence relation defined in that section. As always, $\mathcal{R}$ is a system of rings defined over $\Omega$ with direct sum $R = \oplus\mathcal{R}$. In this section we give a description of all prime ideals of $R$ and of the radical of $R$ for any directed hereditary radical (or arbitrary radical if $\Omega$ has only finitely many idempotents). We do this by passing to the Clifford semigroup $\Omega/\equiv$ and pulling back the results of Section 1.

In order to pass to $\overline{\Omega} = \Omega/\equiv$, we need to define a system $\overline{\mathcal{R}}$ of rings over $\overline{\Omega}$; this is done by assocating to each equivalence class in $\overline{\Omega}$ the direct limit of the rings $R_\alpha$ as $\alpha$ ranges over the equivalence class — see the beginning of Section 4 of [1]. Our situation is a bit simpler than the general case, since if $\alpha \,\eta\, e$, then $e\alpha e \in \Gamma_e$ is a maximal (with respect to $\prec$) element of its $\equiv$-class, and so the direct limit of the rings $R_\alpha$ over that class is just $R_{e\alpha e}$.

We can be even more explicit. Let $E'$ be a subset of $E$ containing exactly one element from each $\eta$-class in $E$, and make $E'$ into a semilattice by defining $e \vee f$ to be the unique element of $E'$ that is $\eta$-equivalent to $ef$. Theorem 2.6 says that we can identify $\Omega/\equiv$ with $\cup_{e\in E'}\Gamma_e$ and implies that $\pi$ restricts to a semilattice isomorphism between $E'$ and the set of idempotents of $\Omega/\equiv$. Thus $\overline{R} = \oplus_{e\in E', x\in\Gamma_e} R_x$.

We will use $\pi$ to denote the natural map from $\Omega$ to $\overline{\Omega} = \Omega/\equiv$ as well as the induced map from $R$ to the direct sum $\overline{R}$ of the system $\overline{\mathcal{R}}$. Thus we have

$$\pi(\bigoplus_{\alpha\in\Omega} r_\alpha) = \bigoplus_{e\in E', x\in\Gamma_e} \sum_{\alpha\equiv x} \phi_{\alpha,x}(r_\alpha).$$

Let us denote the kernel of $\pi : R \to \overline{R}$ by $K = K(\mathcal{R}, \equiv)$. We wish to show $K$ is contained in every prime ideal of $R$; this will enable us to describe the prime ideals and various radicals of $R$ by applying the results of Section 1 to $\overline{R}$. Lemma 4.1 of [1] tells us that $K$ consists of all elements $\oplus_{\alpha\in\Omega} r_\alpha$ such that for any $x \in \cup_{e\in E'}\Gamma_e$, we have $\sum_{\alpha\equiv x} \phi_{\alpha,x}(r_\alpha) = 0$. It is not hard to see that $K$ is generated as an additive subgroup by elements of the form $r_\alpha - \phi_{\alpha,e\alpha e}(r_\alpha)$ for $\alpha \in \Omega$, $e \in E'$, $\alpha\,\eta\,e$.

**Lemma 3.1.** *The ideal $K$ just defined is contained in every prime ideal of $R$.*

*Proof.* Let $P$ be a prime ideal of $R$ and let $\alpha \in \Omega$ and $e \in E$ with $\alpha\,\eta\,e$. We wish to show $r = r_\alpha - \phi_{\alpha,e\alpha e}(r_\alpha) \in P$. Let $I = \oplus_{\beta\in(e)} R_\beta$: plainly $I$ is an ideal of $R$. Lemma 2.4 tells us that $\gamma\alpha\delta = \gamma e\alpha e\delta$ for all $\gamma, \delta \in (e)$. Thus if $r'_\gamma \in R_\gamma$ and $r''_\delta \in R_\delta$, we have

$$r'_\gamma \cdot (r_\alpha - \phi_{\alpha,e\alpha e}(r_\alpha)) \cdot r''_\delta = \\ \phi_{\gamma,\gamma\alpha\delta}(r'_\gamma)\phi_{\alpha,\gamma\alpha\delta}(r_\alpha)\phi_{\delta,\gamma\alpha\delta}(r''_\delta) - \phi_{\gamma,\gamma e\alpha e\delta}(r'_\gamma)\phi_{\alpha,\gamma e\alpha e\delta}(r_\alpha)\phi_{\delta,\gamma e\alpha e\delta}(r''_\delta) = 0.$$

This shows $IrI = 0$.

Since $P$ is prime, this implies that either $r \in P$ or $I \subseteq P$. In the latter case, $e \in \{\, \gamma \in \Omega \mid R_\beta \subseteq P \text{ for all } \beta \in (\gamma) \,\}$. This set is a prime ideal of $\Omega$ by Lemma 2.6(1) of [1]. Since $\alpha\,\eta\,e$, this implies $\alpha$ is also in this ideal, and so $R_\alpha \subseteq P$ and $R_{e\alpha e} \subseteq P$. This implies $r \in P$. ∎

Thus we may pass to $R/K \cong \overline{R}$ to compute prime ideals and radicals containing the prime radical. Before we state our results, we note one more identification. Just as in Section 1, we may replace each $R_{e\alpha e}$ by $R_e$ and obtain $\oplus_{x\in\Gamma_e} R_x \cong R_e[\Gamma_e]$. Thus we may identify $\overline{R}$ with the direct sum of the system of rings over $E'$ with rings $R_e[\Gamma_e]$ and mappings $\phi^{(e,f)} : R_e[\Gamma_e] \to R_f[\Gamma_f]$ defined by $\phi^{(e,f)}(\sum_{x\in\Gamma_e} s_x\, x) = \sum_{x\in\Gamma_e} \phi_{e,f}(s_x) fxf$.

We may now apply the results of Sections 1 and 2 to obtain the following.

**Theorem 3.2.** *Let $\Omega$ be a semigroup having properties (i), (ii), and (iii), with set of idempotents $E$ and with $E'$ defined as above, let $\mathcal{R}$ be a system of rings over $\Omega$, and let $R = \oplus\mathcal{R}$.*

(1) *The prime ideals of $R$ correspond bijectively to prime ideals of the rings $\varinjlim_{e\in E'\setminus\Phi} R_e[\Gamma_e]$ as $\Phi$ ranges over the prime ideals of the semilattice $E'$.*

*Thus the prime ideals in $R$ correspond bijectively to prime families $(\Phi, \mathcal{P})$ with $\Phi$ a prime ideal of the semilattice $E'$ and $\mathcal{P}$ a prime family over the group rings $R_e[\Gamma_e]$ with $e \in E' \setminus \Phi$.*

*Explicitly, this correspondence sends a prime ideal $P$ of $R$ to the pair $(\Phi, \mathcal{P} = (P_e)_{e\in E'\setminus\Phi})$ where $\Phi = \{\, e \in E' \mid R_f \subseteq P \text{ for all } f \in E' \text{ with } fef = f \,\}$ and $P_e$ is the ideal of $R_e[\Gamma_e]$ which corresponds to $\pi(P) \cap \oplus_{\alpha\in\Gamma_e} R_\alpha$. The inverse sends a pair $(\Phi, \mathcal{P})$ to the set of all elements $r = \oplus_{\alpha\in\Omega} r_\alpha \in R$ such that*

$\sum_{\alpha \text{ with } f\in(\alpha)} \phi_{\alpha,f}(r_\alpha) f\alpha f \in P_f$ *for any* $f \in E' \setminus \Phi$ *such that* $f \in (\alpha)$ *for all* $\alpha \in \operatorname{supp} r \setminus \cup_{e\in\Phi}\Omega_e$.

(2) *Let* $\rho$ *be a radical containing the prime radical and assume either* $E'$ *is finite or* $\rho$ *is directed and hereditary. Then*

$$\rho(R) = \{ \oplus_{\alpha\in\Omega} r_\alpha \mid \sum_{\substack{e\in E' \text{ with}\\ fef=f}} \sum_{x\in\Gamma_e} \left( \sum_{\alpha\equiv x} \phi_{\alpha,f}(r_\alpha) \right) fxf \in \rho(R_f[\Gamma_f]) \text{ for all } f \in E' \}$$
$$= \{ \oplus_{\alpha\in\Omega} r_\alpha \mid \sum_{\substack{\alpha \text{ with}\\ f\in(\alpha)}} \phi_{\alpha,f}(r_\alpha) f\alpha f \in \rho(R_f[\Gamma_f]) \text{ for all } f \in E' \}.$$

*Moreover,* $\rho(R) = \{ \oplus_{\alpha\in\Omega} r_\alpha \mid \sum_{\alpha\,\eta\, e} \phi_{\alpha,e}(r_\alpha)\, e\alpha e \in \rho(R_e[\Gamma_e]) \text{ for all } e \in E' \}$ *if and only if* $\phi^{(e,f)}(\rho(R_e[\Gamma_e])) \subseteq \rho(R_f[\Gamma_f])$ *whenever* $e, f \in E'$ *with* $fef = f$. ■

Applying the above and the results of Corollary 1.2, we obtain the following result for semigroup rings.

**Corollary 3.3.** *Let* $\Omega$ *be a semigroup having properties (i), (ii), and (iii), with set of idempotents* $E$ *and with* $E'$ *defined as above, let* $S$ *be a ring, and let* $S[\Omega]$ *be the semigroup ring.*

(1) *The prime ideals of* $S[\Omega]$ *correspond bijectively to prime ideals over the group rings* $S[\varinjlim_{e\in E\setminus\Phi} \Gamma_e]$ *as* $\Phi$ *ranges over the prime ideals of the semilattice* $E'$.

*Thus the prime ideals in* $S[\Omega]$ *correspond bijectively to prime families* $(\Phi, \mathcal{P})$ *with* $\Phi$ *a prime ideal of the semilattice* $E'$ *and* $\mathcal{P}$ *a prime family over the group rings* $S[\Gamma_e]$ *with* $e \in E' \setminus \Phi$, *as in Theorem 3.2*(1).

(2) *Let* $\rho$ *be a radical containing the prime radical and assume either* $E'$ *is finite or* $\rho$ *is directed and hereditary. Then*

$$\rho(S[\Omega]) = \{ \oplus_{\alpha\in\Omega} s_\alpha\, \alpha \mid \sum_{\substack{e\in E' \text{ with}\\ fef=f}} \sum_{x\in\Gamma_e} \left( \sum_{\alpha\equiv x} s_\alpha \right) fxf \in \rho(S[\Gamma_f]) \text{ for all } f \in E' \}$$
$$= \{ \oplus_{\alpha\in\Omega} s_\alpha\, \alpha \mid \sum_{\substack{\alpha \text{ with}\\ f\in(\alpha)}} s_\alpha\, f\alpha f \in \rho(R_f[\Gamma_f]) \text{ for all } f \in E' \}.$$

*Moreover,* $\rho(S[\Omega]) = \{ \oplus_{\alpha\in\Omega} s_\alpha\, \alpha \mid \sum_{\alpha\,\eta\, e} s_\alpha\, e\alpha e \in \rho(R_e[\Gamma_e]) \text{ for all } e \in E' \}$ *if and only if* $f\rho(S[\Gamma_e])f \subseteq \rho(S[\Gamma_f])$ *whenever* $e, f \in E'$ *with* $fef = f$.

(3) *If* $\rho$ *is a radical containing the prime radical and* $\rho(S[G]) = 0$ *for all maximal subgroups* $G$ *of* $\Omega$, *then* $\rho(S[\Omega]) = K$, *the ideal generated by the elements* $s\,e - s\,e\alpha e$ *with* $s \in S$, $\alpha \in \Omega$, $e \in E$, $\alpha\,\eta\, e$. ■

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